AUTOMATIC CONTROL SYSTEMS
(WITH MATLAB PROGRAMS)

S. Hasan Saeed
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*To My Parents*
Properties of Laplace Transforms

1. \( \mathcal{L}\left[ \frac{d}{dt} f(t) \right] = sF(s) - f(0^+) \)

2. \( \mathcal{L}\left[ \frac{d^n}{dt^n} f(t) \right] = s^n F(s) - \sum_{k=0}^{n-1} s^{n-k-1} f^{(k)}(0^+) \)
   where \( \frac{f^{(k)}(t)}{s^{n-k-1}} \) is \( \frac{d^{n-k-1}}{dt^{n-k-1}} f(t) \)

3. \( \mathcal{L}\left[ \int_0^t f(t) \, dt \right] = \frac{F(s)}{s} \)

4. \( \mathcal{L}\left[ \int_{0+}^t f(t) \, dt \right] = \frac{F(s)}{s^2} \)

5. \( \mathcal{L}\left[ e^{-at} f(t) \right] = F(s + a) \)

6. \( \mathcal{L}\left[ t f(t) \right] = -\frac{dF(s)}{ds} \)

7. \( \mathcal{L}\left[ t^n f(t) \right] = (-1)^n \frac{d^n}{ds^n} F(s) \quad n = 1, 2, 3, 4, ... \)

8. \( \mathcal{L}\left[ \frac{1}{t} f(t) \right] = \int_s^\infty F(\zeta) \, d\zeta \) if \( \lim_{t \to 0^+} \frac{1}{t} f(t) \) exists

9. \( \mathcal{L}\left[ \frac{1}{s} f(t) \right] = aF(as) \)

10. \( \mathcal{L}\left[ f_1(t) * f_2(t) \right] = F_1(s) \cdot F_2(s) \) (complex multiplication)

11. Initial value Theorem
    \( \lim_{t \to 0^+} f(t) = \lim_{s \to \infty} s F(s) \)

12. Final value theorem
    \( \lim_{s \to 0^+} s F(s) = \lim_{t \to \infty} f(t) \)
### Laplace Transform Pairs

<table>
<thead>
<tr>
<th>S. No.</th>
<th>$f(t)$</th>
<th>$F(s)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.</td>
<td>Unit Impulse</td>
<td>1</td>
</tr>
<tr>
<td>2.</td>
<td>Unit Step</td>
<td>$1/s$</td>
</tr>
<tr>
<td>3.</td>
<td>$t$</td>
<td>$1/s^2$</td>
</tr>
<tr>
<td>4.</td>
<td>$\frac{t^{n-1}}{(n-1)!}$, $n = 1, 2, 3, \ldots$</td>
<td>$1/s^n$</td>
</tr>
<tr>
<td>5.</td>
<td>$e^{at}$</td>
<td>$\frac{1}{s+a}$</td>
</tr>
<tr>
<td>6.</td>
<td>$te^{at}$</td>
<td>$\frac{1}{(s+a)^2}$</td>
</tr>
<tr>
<td>7.</td>
<td>$\frac{t^{n-1}e^{at}}{(n-1)!}$, $n = 1, 2, 3, \ldots$</td>
<td>$\frac{1}{(s+a)^n}$</td>
</tr>
<tr>
<td>8.</td>
<td>$\frac{e^{at} - e^{bt}}{b-a}$</td>
<td>$\frac{1}{(s+a)(s+b)}$</td>
</tr>
<tr>
<td>9.</td>
<td>$\sin at$</td>
<td>$\frac{\omega}{s^2 + \omega^2}$</td>
</tr>
<tr>
<td>10.</td>
<td>$\cos at$</td>
<td>$\frac{s}{s^2 + \omega^2}$</td>
</tr>
<tr>
<td>11.</td>
<td>$e^{at} \sin at$</td>
<td>$\frac{\omega}{(s+a)^2 + \omega^2}$</td>
</tr>
<tr>
<td>12.</td>
<td>$e^{at} \cos at$</td>
<td>$\frac{s + a}{(s + a)^2 + \omega^2}$</td>
</tr>
<tr>
<td>13.</td>
<td>$\frac{\omega e^{-\frac{\omega t}{\sqrt{1-\xi^2}}}}{\sqrt{1-\xi^2}} \sin \omega t \sqrt{1-\xi^2}$</td>
<td>$\frac{\omega_n^2}{s^2 + 2\xi \omega_n s + \omega_n^2}$</td>
</tr>
<tr>
<td>14.</td>
<td>$\frac{1}{\sqrt{1-\xi^2}} e^{\omega \sqrt{1-\xi^2} t} \sin \left( \omega t - \frac{t}{\sqrt{1-\xi^2}} \right) + \tan^{-1} \left( \frac{\sqrt{1-\xi^2} t}{\xi} \right)$</td>
<td>$\frac{\omega_n^2}{s^2 + 2\xi \omega_n s + \omega_n^2}$</td>
</tr>
<tr>
<td>15.</td>
<td>$\frac{1}{\sqrt{1-\xi^2}} e^{\omega \sqrt{1-\xi^2} t} \sin \left( \omega t - \frac{t}{\sqrt{1-\xi^2}} \right)$</td>
<td>$\frac{\omega_n^2}{s^2 + 2\xi \omega_n s + \omega_n^2}$</td>
</tr>
<tr>
<td>16.</td>
<td>$\frac{1}{ab} \left[ 1 + \frac{1}{a-b} (be^{-at} - ae^{-bt}) \right]$</td>
<td>$\frac{1}{s(s+a)(s+b)}$</td>
</tr>
</tbody>
</table>

### Chapter 1

### Input-Output Relationship

#### 1.1. INTRODUCTION

The control system is very important for all engineers. The first significant control device was James Watt’s flyball governor. This was invented in 1767 to keep the speed of the engine constant by regulating the supply of the steam to the engine.

In control system, the behaviour of the system is described by the differential equations. Minorsky, in 1922 showed that how to determine the stability from the differential equations describing the systems. The differential equations may be ordinary differential equations or the difference equations. The control system can be classified as open loop control system and closed loop control systems.

#### 1.2. OPEN LOOP CONTROL SYSTEM

The open loop control system is also known as control system without feedback or non-feedback control systems. In open loop systems, the control action is independent of the desired output. In this system the output is not compared with the reference input.

The component of the open loop systems are controller and controlled process. The controller may be amplifier, filter etc depends upon the system. An input is applied to the controller and the output of the controller gives to the controlled process & we get the output (desired).

![Fig. 1.1](image)

**Input** → **Controller** → **Controlled process** → **Output**

#### Examples:

1. Automatic washing machine is the example of the open loop systems. In the machine the operating time is set manually. After the completion of set time the machine will stops, with the result we may or may not get the desired (output) amount of cleanliness of washed clothes because there is no feedback is provided to the machine for desired output.

2. Immersion rod is another example of open loop system. The rod heats the water but how much heating is required is not sense by the rod because of no feedback to the rod.

3. A field control d.c. motor is the example of open loop system.

![Fig. 1.2](image)
4. For automatic control of traffic the lamps of three different colours (red, yellow and green) are used. The time for each lamp is fixed. The operation of each lamp does not depend upon the density of the traffic but depends upon the fixed time. Thus, we can say that the control system which operates on the time basis is open loop system.

Advantages:
1. Open loop control systems are simple.
2. Output loop control systems are economical.
3. Less maintenance is required and not difficult.
4. Proper calibration is not a problem.

Disadvantages:
1. Open loop systems are inaccurate.
2. These are not reliable.
3. These are slow.
4. Optimization is not possible.

1.3. CLOSED LOOP CONTROL SYSTEM
Closed loop control systems are also known as feedback control systems. In closed loop control systems the control action is dependent on the desired output. If any system having one or more feedback paths forming a closed loop system.

In closed loop systems the output is compared with the reference input and error signal is produced. The error signal is fed to the controller to reduce the error and desired output is obtained.

Example: In a room we need to regulate the temperature & humidity for comfortable living. Air conditioners are provided with thermostats. By measuring the actual room temperature & compared with desired temperature, an error signal is produced, the thermostat turns ON the compressor or OFF the compressor. The block diagram is shown in fig. 1.4.

---

1.4. COMPARISON BETWEEN OPEN LOOP & CLOSED LOOP
Table 1.1

<table>
<thead>
<tr>
<th>S.No</th>
<th>Open Loop system</th>
<th>Closed Loop System</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>These are not reliable</td>
<td>These are reliable</td>
</tr>
<tr>
<td>2</td>
<td>It is easier to build</td>
<td>It is difficult to build</td>
</tr>
<tr>
<td>3</td>
<td>If calibration is good, they perform accurately</td>
<td>They are accurate because of feedback</td>
</tr>
<tr>
<td>4</td>
<td>Open loop systems are generally more stable</td>
<td>These are less stable</td>
</tr>
<tr>
<td>5</td>
<td>Optimization is not possible</td>
<td>Optimization is possible</td>
</tr>
</tbody>
</table>

1.5. ELEMENTS OR COMPONENTS OF CLOSED LOOP SYSTEMS
The various components of closed loop system are shown in fig. 1.5.

Command: The command is the externally produced input and independent of the feedback control system.

Reference input element: This produces the standard signals proportional to the command.

Error Detector: The error detector receives the measured signal and compares it with reference input. The difference of two signals produces the error signal.

Control element: This regulates the output according to the signal obtained from error detector.

Controller system: This represents what we are controlling by the feedback loop.

Feedback element: This element feedback the output to the error detector for comparison with the reference input.

---

1.6. TRANSFER FUNCTION FOR SINGLE INPUT SINGLE OUTPUT SYSTEM
The transfer function of a linear system is defined as the ratio of Laplace transform of the output to the Laplace transform of the input with all initial conditions zero.

Consider a linear system having input \( r(t) \) & \( c(t) \) is the output of the system, the input-output relation can be described by the following \( n^{th} \)-order differential equation:

\[
d^{n}C(t) + b_{n-1}d^{n-1}C(t) + \ldots + b_{1}dC(t) + b_{0}C(t) = a_{n}d^{n}r(t) + a_{n-1}d^{n-1}r(t) + \ldots + a_{1}dr(t) + a_{0}r(t)
\]

Where \( a' \) & \( b' \) are constants.

Take the Laplace transform of equation (1.1)

\[
a_{n}s^{n}C(s) + b_{n-1}s^{n-1}C(s) + \ldots + b_{1}sC(s) + b_{0}C(s) = a_{n}s^{n}R(s) + a_{n-1}s^{n-1}R(s) + \ldots + a_{1}sR(s) + a_{0}R(s)
\]

We can define the transfer function as

\[
G(s) = \frac{C(s)}{R(s)} = \frac{b_{n}s^{n} + b_{n-1}s^{n-1} + \ldots + b_{1}s + b_{0}}{a_{n}s^{n} + a_{n-1}s^{n-1} + \ldots + a_{1}s + a_{0}}
\]
In equation (1.3), if the order of the denominator polynomial is greater than the order of the numerator polynomial then the transfer function is said to be STRICTLY PROPER. If the order of both polynomials are same, then the transfer function is PROPER. The transfer function is said to be IMPROPER, if the order of numerator polynomial is greater than the order of denominator polynomial.

Consider the block diagram of open loop control system fig. (1.6) where \( R(s) \) & \( C(s) \) are the laplace transform of input and output respectively, then the transfer function \( G(s) \) can be expressed as:

\[
G(s) = \frac{C(s)}{R(s)} = \frac{E_x(t)}{E_r(t)}
\]

\[
(1.4)
\]

1.7. PROCEDURE
The following steps are involved to obtain the transfer function of the given system.

Step 1: Write the differential equations for the given system.

Step 2: Take the laplace transform of the equations obtained in step1, with assumption, all initial conditions are zero.

Step 3: Take the ratio of transformed output to input

Step 4: The ratio of transformed output to the input, obtained in step 3 is the required transfer function of the given system.

1.8. CHARACTERISTIC EQUATION OF A TRANSFER FUNCTION
The characteristic equation of a linear system can be obtained by equating the denominator polynomial of the transfer function to zero. Thus, the characteristic equation of the transfer function of equation 1.3 will be

\[
a_0s^n + a_1s^{n-1} + \ldots + a_0 = 0
\]

(1.5)

1.9. POLES AND ZEROS OF A TRANSFER FUNCTION
Consider equation 1.3, the numerator & denominator can be factored in \( m \) and \( n \) terms respectively, then the equation 1.3 can be expressed as

\[
\frac{C(s)}{R(s)} = \frac{K(s + Z_1)(s + Z_2)\ldots(s + Z_n)}{(s + P_1)(s + P_2)\ldots(s + P_m)}
\]

\[
(1.6)
\]

Where \( K = \frac{b_0}{a_0} \) is known as the gain factor, \( s \) is the complex frequency

POLES: The poles of \( G(s) \) are those values of \( s \) which make \( G(s) \) tend to infinity. For example in equation 1.6 we have poles at \( s = -P_1, s = -P_2 \) and a pair of poles at

\[
s = -\frac{b \pm \sqrt{b^2 - 4ac}}{2a}
\]

(1.7)

ZEROS: The zeros of \( G(s) \) are those values of \( s \) which make \( G(s) \) tend to zero. For example in eq. 1.6 we have zeros at \( s_1 = Z_1, s_2 = Z_2 \) and a pair of zeros at

\[
s = -\frac{b \pm \sqrt{b^2 - 4ac}}{2a}
\]

(1.8)

If either poles or zeros coincide, then such type of poles or zeros are called multiple poles or the repetitive factor in denominator and multiple zeros are due to the repetitive factor in numerator of a transfer function.

For example, consider the transfer function

\[
G(s) = \frac{50(s + 3)}{s(s + 2)(s + 4)^2}
\]

(1.9)

The above transfer function having the simple poles at \( s = 0, s = -2, \) multiple poles at \( s = -4 \) i.e. the pole of order 2 at \( s = -4 \) and simple zero at \( s = -3 \).

The above mentioned poles and zeros are of finite values. If we consider the entire \( s \)-plane including infinity then two cases arise.

1. If the no. of zeros are less than the no. of poles i.e. \( Z < P \) then the value of the transfer function becomes zero for \( s \to \infty \). Hence we can say that there are zeros at infinity and the order of such zeros is \( P - Z \). For example in equation 1.9 there are four finite poles at \( s = 0, -2, -4 \) and there is one finite zero at \( s = -3 \) but there are three zeros at infinity \( P - Z = 4 - 1 = 3 \). Therefore, the function has a total of four poles and four zeros in the entire \( s \)-plane including infinity.

2. If the no. of poles are less than the no. of zeros \( P < Z \) then the value of the transfer function becomes infinity for \( s \to \infty \). Hence we can say there are poles at infinity \( (s \to \infty) \) and the order of the poles will be \( Z - P \).

Therefore, in addition to finite poles and zeros, if we consider poles and zeros at infinity, then for a rational function the total number of zeros will be equal to the total number of poles.

The pole is represented by 'X' and zero by 'O'. These symbols are used to locate the poles and zeros on \( s \)-plane. The pole-zero plot of eq. 1.9 is shown in fig. 1.7.

1.10. IMPULSE RESPONSE
From equation 1.4

\[
C(t) = R(t) \times G(s)
\]

Suppose, a system is subjected to a unit impulse, then the output will be

\[
C(t) = G(s)
\]

because the Laplace transform of unit impulse function is unity.

The inverse laplace of eq. 1.11 will be

\[
C(t) = g(t)
\]

(1.12)

where \( g(t) \) is the unit impulse response of a system. Therefore inverse laplace of \( G(s) \) is called impulse response or the transfer function of a system is the laplace transform of its impulse response.

Practically, it is not possible to generate a true impulse. A pulse with less duration than the time constant of the system can be considered as an impulse & denoted by \( \delta(t) \).
Example 1.1. Find the transfer function of the given network
Solution: Step 1: Apply KVL in mesh (1)
   \[ V_i = R_i + \frac{di}{dt} \]  \( \ldots \) (1.13)
   \[ V_i = R_i + sL \frac{di}{dt} \]  \( \ldots \) (1.15)
   \[ V_o = L \frac{di}{dt} \]  \( \ldots \) (1.14)
   \[ V_o = sL \frac{di}{dt} \]  \( \ldots \) (1.16)

Step 2: Take laplace transform of equation 1.13 & 1.14 with assumption that all initial conditions are zero.
   \[ \frac{V_o}{V_i} = \frac{sL}{R + sL} \]  \( \ldots \) (1.17)

Step 3: Calculation of transfer function
   \[ \frac{V_o}{V_i} = \frac{sL}{R + sL} \]  \( \ldots \) (1.17)

Eq. 1.17 is the required transfer function.

Example 1.2. Determine the transfer function of the electrical network shown in fig. 1.10.
Solution: Step 1: Apply KVL in both meshes
   \[ E_i = R_i + L \frac{di}{dt} + \frac{1}{C} \int i dt \]  \( \ldots \) (1.18)
   \[ E_o = \frac{1}{C} \int i dt \]  \( \ldots \) (1.19)

Step 2: Take laplace transform of eqn 1.18 & 1.19
   \[ E_i(s) = R_i(s) + sL_i(s) + \frac{1}{Cs} i(s) \]  \( \ldots \) (1.20)
   \[ E_i(s) = \frac{1}{Cs} i(s) \]  \( \ldots \) (1.21)

Step 3: Determination of transfer function
   \[ V_o(s) \left[ R_1 + R_2 + R_3 C_s \right] = V_i(s) \frac{1 + R_1 C_s}{R_1} \]  \( \ldots \) (1.22)

Example 1.3. Obtain the transfer function \[ \frac{V_2(s)}{V_1(s)} \] for fig. 1.11.
Solution: Step 1: KCL at node 'a'
   \[ i = i_1 + i_2 \]  \( \ldots \) (1.23)
   \[ i_1 = \frac{V_1 - V_2}{R_1} \]  \( \ldots \) (1.24)
   \[ i_2 = C \frac{d}{dt} (V_1 - V_2) \]  \( \ldots \) (1.25)
   \[ i = i_2 = \frac{V_2}{R_2} \]  \( \ldots \) (1.26)

Step 2: Take laplace transform of eqn 1.24
   \[ \frac{V_2}{R_2} = \frac{1}{R_1} V_1(s) - \frac{1}{R_1} V_2(s) + Cs V_1(s) - Cs V_2(s) \]  \( \ldots \) (1.27)

Step 3: Determination of transfer function
   \[ \frac{V_2(s)}{V_1(s)} = \frac{R_1 + R_2 + R_3 C_s}{R_1 R_2 R_3} \]  \( \ldots \) (1.28)

Example 1.4. Find the transfer function of lag network shown in fig. 1.12.
Solution: Step 1: Apply KVL in both meshes
   \[ e_i(t) = R_1 i(t) + R_2 \frac{1}{C} \int i(t) dt \]  \( \ldots \) (1.29)
   \[ e_i(t) = \frac{1}{R_2} \int i(t) dt \]  \( \ldots \) (1.30)
Step 2: Laplace transform of \( e_0^a \) (1.26) & (1.27)

\[
E_0(s) = \left( \frac{R_1 + R_2 + \frac{1}{C_2}}{R_1 C_s + R_2 C_s + 1} \right) I(s)
\]

\[
E_0(s) = \left( \frac{R_2 + \frac{1}{C_2}}{R_2 C_s + 1} \right) I(s)
\]

Step 3: Calculation of transfer function

\[
E_0(s) = \left( \frac{1 + R_2 C_s}{R_2 C_s + R_1 C_s + 1} \right) I(s)
\]

Equation 1.28 is the required transfer function.

Example 1.5. Determine the transfer function of fig 1.13.

Solution: Step 1: Calculation of \( Z_1 \):

\[
Z_1 = \frac{R_1 s C_1}{R_1 + \frac{1}{s C_1}} = \frac{R_1}{R_1 C_s + 1}
\]

...(1.29)

Step 2: Calculation of \( Z_2 \):

\[
Z_2 = R_2 + \frac{1}{s C_2}
\]

...(1.30)

Step 3: Calculation of transfer function in terms of \( Z_1 \) & \( Z_2 \)

\[
\frac{E_0(s)}{E_i(s)} = \frac{Z_2(s)}{Z_2(s) + Z_2(s)}
\]

...(1.31)

Step 4: Calculation of transfer function in terms of \( R_1, R_2, C_1, \) & \( C_2 \) Put the values of \( Z_1(s) \) & \( Z_2(s) \)

from eq 1.29 & 1.30 in eq 1.31

\[
\frac{E_0(s)}{E_i(s)} = \frac{(1 + R_2 C_s) C_2}{R_2 C_s + 1}
\]

The above \( e_0^a \) is the required transfer function of the given circuit.

Example 1.6. Determine the transfer function of given transformer coupled circuit (fig. 2.9).

Solution: Apply KVL in both meshes

\[
e_i(t) = \frac{R_1 i_1(t)}{C_1} + \int \frac{i_1(t) dt}{C_1} + \frac{R_2 i_2(t)}{C_2}
\]

...(1.33)

\[
e_2(t) = R_2 i_2(t)
\]

...(1.34)

\[
0 = R_2 i_2(t) + i_1(t) + \frac{L_2}{C_2} \frac{d i_2(t)}{dt} + \frac{1}{C_2} \int i_2(t) dt - M \frac{d i_1(t)}{dt}
\]

...(1.35)

Take Laplace transform of \( e_0^a \) 1.33, 1.34 & 1.35

\[
E_1(s) = i_1(s) + \frac{R_1}{C_1} s + \frac{R_2}{C_2} s + \frac{S L}{C_2}
\]

...(1.36)

\[
E_2(s) = i_2(s)
\]

...(1.37)

\[
0 = i_2(s) [- R_2 + s(L_2 + L_1) + 1/s C_2] - s M I_1(s)
\]

...(1.38)

Solving the equations 1.36, 1.37 & 1.38 the required transfer function.

\[
G(s) = \frac{S^2 R_2 R_2 C_2 M}{S^2 R_2 C_2 + S^2 C_2 (L_2 + L_1) + \frac{1}{C_2} S^2 L_2 C_1 + SC_1 R_1 + 1 - M^2 S^2 C_2} \quad \text{Ans.}
\]

1.11. TRANSLATIONAL SYSTEMS

The motion takes place along a straight line is known as translational motion. There are three types of forces that resist motion.

1. Inertia Force: Consider a body of mass 'M' & acceleration 'a', then according to Newton's second law of motion the inertia force will be equal to the product of mass 'M' & acceleration 'a'.

\[
F_M(t) = M a(t)
\]

...(1.39)

In terms of velocity the eq 1.39 becomes

\[
F_M(t) = M \frac{dv(t)}{dt}
\]

...(1.40)

In terms of displacement the eq 1.39 can be expressed as

\[
F_M(t) = M \frac{d^2 x(t)}{dt^2}
\]

...(1.41)

Fig 1.16.
2. Damping Force: For viscous friction we assume that the damping force is proportional to the velocity.

\[ F_d(t) = B \dot{x}(t) = B \frac{dx}{dt} \]  

Where, \( B \) = Damping coefficient

unit of \( B = N\cdot m/\text{sec} \).

We can represent \( B \) by a dashpot, consists of piston and cylinder.

3. Spring Force: A spring stores the potential energy. The restoring force of a spring is proportional to the displacement.

\[ F_s(t) = \frac{d}{dt}(Kx(t)) \]  

Where \( K \) = spring constant or stiffness

unit of \( K = N/m \).

The stiffness of a spring can be defined as restoring force per unit displacement.

### 1.12. ROTATIONAL SYSTEM

The rotational motion of a body can be defined as the motion of a body about a fixed axis. There are three types of torques resisting the rotational motion.

1. Inertia Torque: Inertia \( (I) \) is the property of an element that stores the kinetic energy of rotational motion. The inertia Torque \( T_I \) is the product of moment of Inertia \( I \) and angular acceleration \( \alpha(t) \).

\[ T_I(t) = I \alpha(t) \]

\[ T_I(t) = \frac{d}{dt} \omega(t) \]

\[ T_I(t) = \frac{d^2}{dt^2} \theta(t) \]  

Where \( \omega(t) \) = angular velocity

\( \theta(t) \) = angular displacement

unit of Torque = \( N\cdot m\).

2. Damping Torque: The damping Torque \( T_d(t) \) is the product of damping coefficient \( B \) and angular velocity \( \dot{\theta}(t) \).

\[ T_d(t) = B \dot{\theta}(t) \]  

\[ T_d(t) = B \frac{d\theta}{dt} \]  

3. Spring Torque: Spring torque \( T_s(t) \) is the product of torsional stiffness and angular displacement. \( K \) is a constant of torsional stiffness.

\[ T_s(t) = KB\dot{\theta}(t) \]

unit of \( K = N\cdot m/\text{rad} \).

If we compare the equations, we get an analogous system. The analogous quantities are tabulated as follows:

<table>
<thead>
<tr>
<th>S.No.</th>
<th>Translational</th>
<th>Rotational</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.</td>
<td>Force, ( F )</td>
<td>Torque, ( T )</td>
</tr>
<tr>
<td>2.</td>
<td>Acceleration, ( a )</td>
<td>Angular acceleration, ( \alpha )</td>
</tr>
<tr>
<td>3.</td>
<td>Velocity, ( v )</td>
<td>Angular velocity, ( \omega )</td>
</tr>
<tr>
<td>4.</td>
<td>Displacement, ( x )</td>
<td>Angular displacement, ( \theta )</td>
</tr>
<tr>
<td>5.</td>
<td>Mass, ( M )</td>
<td>Moment of inertia, ( I )</td>
</tr>
<tr>
<td>6.</td>
<td>Damping coefficient ( B )</td>
<td>Rotational damping coeff., ( B )</td>
</tr>
<tr>
<td>7.</td>
<td>Stiffness</td>
<td>Torsional stiffness</td>
</tr>
</tbody>
</table>

1.13. **D'ALEMBERT'S PRINCIPLE**

This principle states that "for any body, the algebraic sum of externally applied forces and the forces resisting motion in any given direction is zero".

D'Alembert's principle is useful in writing the equation of motion of mechanical systems. Consider a system shown in fig. 1.21 consisting of a mass \( M \), spring & dashpot.

First choose a reference direction. All the forces in the direction of reference direction considered as positive & the forces oppose to the reference direction taken as negative.

**External Force:** \( F(t) \)

**Resisting Force:** a. Inertia Force \( F_m(t) = -M\frac{d^2x}{dt^2} \)

b. Damping Force \( F_d(t) = -B\frac{dx}{dt} \)

c. Spring Force \( F_s(t) = -Kx(t) \)

\[ F(t) = F_m(t) + F_d(t) + F_s(t) = 0 \]

\[ F(t) = M\frac{d^2}{dt^2}x(t) - B\frac{dx}{dt} + Kx(t) \]  

or, \[ F(t) = M\frac{d^2}{dt^2}x(t) + B\frac{dx}{dt} + Kx(t) \]

Consider the rotational system shown in fig. 1.22.

**External Torque:** \( T(T) \)

**Resisting Torque:** a. Inertia Torque \( T_I(t) = -I\frac{d\omega(t)}{dt} \)

b. Damping Torque \( T_d(t) = -B\frac{d\theta(t)}{dt} \)

c. Spring Torque \( T_s(t) = -K\theta(t) \)

According to D'Alembert principle

\[ T(t) = T_I(t) + T_d(t) + T_s(t) = 0 \]

\[ T(t) = I\frac{d^2\omega(t)}{dt^2} - B\frac{d\theta(t)}{dt} - K\theta(t) = 0 \]
Example 1.8. Write the differential equations describing the dynamics of the system shown in Fig. 1.25 and find the ratio \( \frac{X_2(s)}{F(s)} \).

Solution: Free body diag. For mass \( M_1 \):

\[
F(t) = M_1 \frac{d^2}{dt^2} x_1 + K_1(x_1 - x_2) = M_1 \frac{d^2 x_1}{dt^2} + K_1(x_1 - x_2)
\]

\[
F(t) = M_1 \frac{d^2}{dt^2} x_1 + K_1(x_1 - x_2) = M_1 \frac{d^2 x_1}{dt^2} + K_1(x_1 - x_2)
\]

\[
K_1 (x_1 - x_2) = K_2 (x_1 - x_2)
\]

\[
K_1 (x_1 - x_2) = K_2 (x_1 - x_2)
\]

Take laplace transform of eqn \(1.51\), assume initial conditions zero.

\[
F(s) = M_1 s^2 X_1(s) + K_1 X_1(s) - K_2 X_2(s)
\]

\[
F(s) = M_1 s^2 X_1(s) + K_1 X_1(s) - K_2 X_2(s)
\]

\[
X_1(s) = \frac{X_1(s)}{K_1}
\]

\[
X_1(s) = \frac{X_1(s)}{K_1}
\]

Put the value of \( X_1(s) \) in eqn \(1.53\).

\[
F(s) = \frac{X_2(s)}{K_1} \left( s^2 M_2 + K_1 + K_2 \right) - K_1 X_2(s)
\]

\[
F(s) = \frac{X_2(s)}{K_1} \left( s^2 M_2 + K_1 + K_2 \right) - K_1 X_2(s)
\]

\[
\frac{X_1(s)}{F(s)} = \frac{X_1(s)}{\left( s^2 M_2 + K_1 + K_2 \right) - K_1^2}
\]

\[
\frac{X_1(s)}{F(s)} = \frac{X_1(s)}{\left( s^2 M_2 + K_1 + K_2 \right) - K_1^2}
\]
Example 1.9. A mass spring system under equilibrium condition is shown in fig. 1.27. Drive the system equation where $M = 10$ kg. $P = 30 \text{ N}/\text{m/sec.}$ & $K = 20 \text{ N/m}.$

Solution: Free body diag.

Differential equation

$$F = M \frac{d^2x}{dt^2} + B \frac{dx}{dt} + Kx$$

Put the given values

$$F = 10 \frac{d^2x}{dt^2} + 30 \frac{dx}{dt} + 20x$$

Ans. \hspace{1cm} \ldots(1.55)

Example 1.10. Drive the system equations & find the value of $X_2(s)/F(s)$ for the system shown in fig. 1.29.

Solution: Free body diag. for mass $M_1$.

Free body diag. For mass $M_2$.

System equation for mass $M_1$:

$$F(t) = M_1 \frac{d^2x_1}{dt^2} + f_1 \frac{dx_1}{dt} + B_{12} \frac{d}{dt}(x_1 - x_2) + K_{12}x_2$$

\hspace{1cm} \ldots(1.56)

System equation for mass $M_2$:

$$B_{12} \frac{d}{dt}(x_1 - x_2) = M_2 \frac{d^2x_2}{dt^2} + f_2 \frac{dx_2}{dt} + K_{22}x_2$$

\hspace{1cm} \ldots(1.57)

Laplace transform of equation 1.56

$$F(s) = X_1(s)\left(s^2M_1 + B_{12}s + f_1 + K_{12}\right) - B_{12}sX_2(s)$$

Laplace transform of equation 1.57

$$B_{12}X_1(s) = (s^2M_2 + B_{12}s + f_2 + K_{22})X_2(s)$$

\hspace{1cm} \ldots(1.58)

1.15. ANALOGOUS SYSTEM

Consider a series RLC circuit. Apply Kirchhoff's voltage law

$$E = Ri + L \frac{di}{dt} + \frac{1}{C} \int idt$$

\hspace{1cm} \ldots(1.60)

In terms of charge $q$, 1.60 becomes

$$E = R \frac{d}{dt} \frac{1}{C} \frac{dq}{dt}$$

\hspace{1cm} \ldots(1.61)

Now consider a parallel RLC circuit. Now Kirchhoff's current law

$$I = E \frac{1}{L} \int (E dt + C \frac{dE}{dt})$$

\hspace{1cm} \ldots(1.62)

In terms of magnetic flux linkage, the eqn. 1.62 becomes

$$\phi = \int (E dt)$$

Since

$$I = \frac{1}{L} \left( \frac{d\phi}{dt} \right) + \frac{1}{C} \frac{dE}{dt}$$

\hspace{1cm} \ldots(1.63)

Now, compare the equation 1.47 with the equation 1.61, both equations are differential equations of same order i.e. identical such type of systems whose differential equations are in the same form are called analogous systems.

On comparison the eqn. 1.47 with eqn. 1.61 we can see that in mechanical system the force $F$ is analogous to voltage $E$ in electrical system, such type of analogy is known as force-voltage analogy.

From equation 1.47 & 1.61, mass ($M$) is analogous to inductance ($L$), coeff. of viscous friction ($B$) is analogous to resistance ($R$), spring stiffness ($K$) is analogous to $\frac{1}{C}$ & so on.

Table 1.3

<table>
<thead>
<tr>
<th>S.No.</th>
<th>Mechanical Translational System</th>
<th>Electrical System</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.</td>
<td>Force ($F$)</td>
<td>Voltage ($E$)</td>
</tr>
<tr>
<td>2.</td>
<td>Mass ($M$)</td>
<td>Inductance ($L$)</td>
</tr>
<tr>
<td>3.</td>
<td>Stiffness ($K$)</td>
<td>Reciprocal of capacitance ($\frac{1}{C}$)</td>
</tr>
<tr>
<td>4.</td>
<td>Damping coeff. ($B$)</td>
<td>Capacitance ($C$)</td>
</tr>
<tr>
<td>5.</td>
<td>Displacement ($x$)</td>
<td>Resistance ($R$)</td>
</tr>
<tr>
<td>6.</td>
<td></td>
<td>Charge ($q$)</td>
</tr>
</tbody>
</table>
Now compare the equation (1.47) with eqn (1.63). Since the force \( F \) is analogous to the current source, such type of analogy is known as force-current analogy. The analogous quantities are tabulated as:

### Table 1.4

<table>
<thead>
<tr>
<th>S.No.</th>
<th>Mechanical Translational System</th>
<th>Electrical system</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.</td>
<td>Force ( (F) )</td>
<td>Current ( (I) )</td>
</tr>
<tr>
<td>2.</td>
<td>Mass ( (M) )</td>
<td>Capacitance ( (C) )</td>
</tr>
<tr>
<td>3.</td>
<td>Damping coeff. ( (B) )</td>
<td>Reciprocal of resistance ( (1/R) )</td>
</tr>
<tr>
<td>4.</td>
<td>Stiffness ( (K) ) (Elastance, ( 1/K ))</td>
<td>Reciprocal of inductance ( (1/L) )</td>
</tr>
<tr>
<td>5.</td>
<td>Displacement ( (x) )</td>
<td>Inductance ( (L) )</td>
</tr>
<tr>
<td>6.</td>
<td>Velocity ( (x) )</td>
<td>Flux linkage ( (\phi) )</td>
</tr>
</tbody>
</table>

Now consider the rotational system & compare the eqn 1.48 with 1.61. On comparison the analogous quantities can be tabulated as:

### Table 1.5: Torque-voltage analogy

<table>
<thead>
<tr>
<th>S.No.</th>
<th>Mechanical Rotational System</th>
<th>Electrical system</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.</td>
<td>Torque ( (T) )</td>
<td>Voltage ( (E) )</td>
</tr>
<tr>
<td>2.</td>
<td>Moment of Inertia ( (I) )</td>
<td>Inductance ( (L) )</td>
</tr>
<tr>
<td>3.</td>
<td>Damping coeff. ( (B) )</td>
<td>Resistance ( (R) )</td>
</tr>
<tr>
<td>4.</td>
<td>Stiffness ( (K) ) (Elastance, ( 1/K ))</td>
<td>Reciprocal of capacitance ( (1/C) )</td>
</tr>
<tr>
<td>5.</td>
<td>Angular displacement ( (\theta) )</td>
<td>Charge ( (Q) )</td>
</tr>
<tr>
<td>6.</td>
<td>Angular Velocity ( (\omega) )</td>
<td>Current ( (I) )</td>
</tr>
</tbody>
</table>

On comparison the eqn 1.48 with eqn 1.63 we get Torque-current analogy and can be tabulated as:

### Table 1.6

<table>
<thead>
<tr>
<th>S.No.</th>
<th>Mechanical Rotational System</th>
<th>Electrical System</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.</td>
<td>Torque ( (T) )</td>
<td>Current ( (I) )</td>
</tr>
<tr>
<td>2.</td>
<td>Moment of Inertia ( (I) )</td>
<td>Capacitance ( (C) )</td>
</tr>
<tr>
<td>3.</td>
<td>Damping coeff. ( (B) )</td>
<td>Reciprocal of resistance ( (1/R) )</td>
</tr>
<tr>
<td>4.</td>
<td>Stiffness ( (K) ) (Elastance, ( 1/K ))</td>
<td>Reciprocal of inductance ( (1/L) )</td>
</tr>
<tr>
<td>5.</td>
<td>Angular displacement ( (\theta) )</td>
<td>Inductance ( (L) )</td>
</tr>
</tbody>
</table>

Following Cheng*, the rule for drawing the analogous circuit can be stated as:

(a) For force-Voltage analogy: Each junction in the mechanical system corresponds to a closed driving sources and passive elements connected to the junction. All points on a rigid mass are considered as the junction.

For force-current analogy: Each junction in the mechanical system corresponds to a node which joins electrical excitation sources and passive elements analogous to the mechanical driving sources and passive elements connected to the junction. All points on a rigid mass are considered as the same junction and one terminal of the capacitance analogous to a mass is always connected to the ground.

The reason for connecting one terminal of capacitance analogous to a mass is always connected to the ground is that the velocity (or displacement) of a mass is always referred to the earth.

Consider the system shown in fig. 1.21, corresponding to \( x \) in mechanical system, we will have one loop in \( f-V \) analogous electrical circuit. The circuit consists of a voltage source, inductance \( L(M) \), resistance \( R(B) \), the reciprocal of capacitance \( (1/C) \) and charge \( (Q) \).

The equation for the mechanical system shown in fig. 1.21 is

\[
f = M \frac{d^2x}{dt^2} + B \frac{dx}{dt} + Kx \tag{1.64}
\]

For the electrical circuit the equation will be

\[
V = \frac{d^2q}{dt^2} \tag{1.65}
\]

Since, \( q = \int i dt \), the equation (1.65) will becomes

\[
V = L \frac{di}{dt} + Ri + \frac{1}{C} \int i dt \tag{1.66}
\]

The equation (1.66) is the voltage equation for series RL circuit.

Series RL circuit, is the analogous electrical circuit of the mechanical system shown in fig. 1.21.

**Example 1.11.** Draw the analogous electrical network of the given system.

**Solution:** Now corresponding \( x_1, x_2, x_3 \) we have three loops in electrical circuit.

First loop consists of a voltage source \( V(f) \), inductance \( L_1(M_1) \), resistance \( R_1 \), the resistance \( R_1 \) is common in between first & second loop.

Second loop consists of inductance \( L_2(M_2) \), resistance \( R_2(B_2) \) and \( \frac{1}{C_2} \).

Resistance \( R_2 \) & \( \frac{1}{C_2} \) are common in loop second & third loop.

Similarly, third loop consists of inductance \( L_3(M_3) \), \( \frac{1}{C_3} \) & \( R_3(B_3) \),

\( q_1, q_2 \) & \( q_3 \) are analogous quantities of \( x_1, x_2 \) & \( x_3 \) respectively.

* From David K. Cheng, Analysis of linear systems, Addison-Wesley
Example 1.12. Draw the electrical analogous circuit of the system shown in Fig 1.35.

Solution: By inspection & with the help of table we can draw the electrical circuit shown in Fig 1.36.

Example 1.13. Draw the analogous electrical circuit of the system shown in Fig. 1.37. Use f-α analogy.

Solution: Corresponding $x_1$ & $x_2$, we have two loops in electrical circuit. The analogous electrical circuit shown in Fig. 1.38.

Example 1.14. Draw the analogous electrical circuit of the given system shown in Fig. 1.37 use f-α analogy. Assume frictionless wheels.

Solution: Since, wheels are frictionless, $B_1 = 0$ & $B_2 = 0$

Corresponding $x_1$ & $x_2$, we have two independent nodes. The analogous electrical circuit shown in Fig. 1.39.

1.16. MECHANICAL EQUIVALENT NETWORK

The equivalent mechanical network is useful to determine the transfer function of the system. The following procedure is applied to draw the mechanical equivalent network.

Step 1: Draw a reference line. Consider the fixed points as reference.
Step 2: Corresponding to the displacement $x_1, x_2$, select the nodes.

Step 3: Connect one end of masses to the reference line.

Step 4: Connect other elements of the system to the nodes.

Step 5: Apply nodal analysis, write the system equations.

Consider a simple system shown in fig. 1.21. The mechanical equivalent network is shown in fig. 1.43.

Apply the nodal analysis the system equation will be

$$ f(t) = M_1 \frac{d^2 x_1}{dt^2} + B_1 \frac{dx_1}{dt} + K_1 x_1 $$

With the help of mechanical equivalent network, we can draw the analogous electrical circuit (shown in fig. 1.44).

Example 1.17. Draw the mechanical network of the system shown in Fig. 1.37.

Solution: Since, there are two displacements $x_1, x_2$ we have two nodes.

Example 1.18. Draw the mechanical equivalent network, write the system equations and find $X_1(t)$ of the system shown in Fig. 1.46.

Solution: Mechanical network is shown in fig. 1.47.

Apply nodal analysis at $x_1$ & $x_2$

$$ f(t) = M_1 \frac{d^2 x_1}{dt^2} + B_1 \frac{dx_1}{dt} + K_1 x_1 $$

$$ f(t) = M_2 \frac{d^2 x_2}{dt^2} + B_2 \frac{dx_2}{dt} + K_2 x_2 $$

$$ K_1 (x_1 - x_2) + M_1 \frac{d^2 x_1}{dt^2} + K_2 x_2 = 0 $$

Laplace transform of eqn. 1.68 & 1.69

$$ F(s) = s^2 M_1 X_1(s) + (K_1 + K_2) X_1(s) - K_1 X_2(s) $$

$$ s^2 M_2 X_2(s) + (K_2 + K_3) X_2(s) - K_2 X_1(s) = 0 $$

Form eqn. 1.71

$$ X_1(s) = \frac{s^2 M_1 + K_1 + K_2}{K_2} X_2(s) $$

Put the value of $X_1(s)$ from 1.72 in eqn 1.70 & solve for $\frac{F_1(s)}{X_2(s)}$

$$ \frac{F_1(s)}{X_2(s)} = \frac{(s^2 M_1 + K_1 + K_2)(s^2 M_2 + K_2 + K_3) - K_2^2}{K_2} $$

Eqn. 1.73 is the required answer.

1.17. MECHANICAL COUPLING

In fig. 1.48 two wheels are mechanically coupled. At the points $A$ and $B$ wheels experiences equal and opposite forces and the linear velocity at the point of contact will be same.

Let $\omega_1$ & $\omega_2$ = angular velocities

$T_1$ & $T_2$ = Torque

$r_1$ & $r_2$ = radius of the wheels

$\theta_1$ & $\theta_2$ = angular displacements

$N_1$ & $N_2$ = No. of teeth on wheels

In ideal case work done by wheel (1) will be equal to the work done by wheel (2)

Work done by wheel (1) = $T_1 \theta_1$

Work done by wheel (2) = $T_2 \theta_2$

$$ \frac{T_1}{T_2} = \frac{\theta_2}{\theta_1} $$

$T_2 = T_3 \theta_1$

$$ T_1 \theta_1 = T_3 \theta_2 $$

$$ T_1 \theta_1 = T_3 \theta_2 $$

$$ \frac{T_1}{T_3} = \frac{\theta_2}{\theta_1} $$

$$ \frac{T_1}{T_3} = \frac{\theta_2}{\theta_1} $$
Linear distance travelled along the surface of wheel (1) = \( \theta_1 r_1 \)
Linear distance travelled along the surface of wheel (2) = \( \theta_2 r_2 \)
The linear distance will be same
\[ \theta_1 r_1 = \theta_2 r_2 \]
\[ \frac{\theta_1}{\theta_2} = \frac{r_1}{r_2} \]
Combining the equation 1.74 & 1.75
\[ \frac{r_1}{r_2} = \frac{\theta_1}{\theta_2} = \frac{N_1}{N_2} \]
\[ N_1 \alpha r_1 \]
\[ N_2 \alpha r_2 \]
No. of teeth on wheels \( \times \) radius of wheels
\[ N_1 \alpha r_1 \]
\[ N_2 \alpha r_2 \]
Combining the equations 1.76, 1.77 & 1.78
\[ \frac{r_1}{r_2} = \frac{\theta_1}{\theta_2} = \frac{N_1}{N_2} \]
Since,
\[ \frac{\theta_1}{\theta_2} = \frac{r_1}{r_2} \]
\[ \frac{\theta_1}{\theta_2} = \frac{N_1}{N_2} \]
Differentiate eqn 1.80
\[ \frac{\theta_1}{\theta_2} = \frac{r_1}{r_2} \]
But
\[ \frac{\theta_1}{\theta_2} = \omega_1 \]
\[ \frac{\theta_2}{\theta_1} = \omega_2 \]
Combining the eqn 1.79 & 1.81
\[ \frac{\theta_1}{\omega_1} = \frac{r_1}{\omega_1} = \frac{N_1}{N_2} \]
If we consider \( r_1 : r_2 \) as the turn ratio of an ideal transformer, then angular velocity will be analogous to current and torque will be analogous to voltage. This is \( f-V \) analogy.

In case of \( f-I \) analogy

\[ I_1(T_1) \]
\[ I_2(T_2) \]
\[ V_1(\omega_1) \]
\[ V_2(\omega_2) \]
\[ N_1' : N_2' \text{ is the turn ratio of the transformer} \]

Fig. 1.50.

Example 1.19.
(a) Find the ratio of \( r_1 : r_2 \)
(b) If \( \omega_1 = 10 \text{ rad/sec} \), calculate \( \omega_2 \)
(c) If torque for wheel (1) is 5 Nm, What will be the torque for wheel (2)
(d) If the displacement of wheel (1) is 20°. What is the displacement of wheel (2).

Solution : We know that
\[ \frac{n_1}{n_2} = \frac{N_1}{N_2} \]
\[ \frac{N_1}{N_2} = \frac{40}{20} = \frac{2}{1} \]
\[ \frac{n_1}{n_2} = \frac{2}{1} \]
\[ \omega_2 = 20 \text{ rad/sec} \]
\[ \frac{N_1}{N_2} = \frac{40}{20} = \frac{2}{1} \]
\[ \omega_2 = 20 \text{ rad/sec} \]
\[ \frac{T_1}{T_2} = \frac{5}{20} \]
\[ \frac{N_1}{N_2} = \frac{40}{20} \]
\[ \frac{T_1}{T_2} = \frac{2.5 \text{ Nm}}{\omega_2 = 40^0} \]
Example 1.20.
(a) If \( \theta_1 = 3 \text{ rad. (clockwise)} \) calculate the displacement of wheel 2, 3 and 4.
(b) If \( \omega_1 = 15 \text{ rad/sec} \). Calculate \( w_2 \) & \( \omega_4 \).
(c) If \( T_2 = 10 \text{ Nm} \). Calculate \( T_2 \) & \( T_4 \).
(d) If angular acceleration \( (\ddot{\theta}) \) for wheel (1) is \( 4 \text{ rad/sec}^2 \) calculate \( \ddot{\theta}_1 \).
(e) Find \( r_1 : r_2 \) & \( r_1 : r_3 \).

Solution:

\[
\begin{align*}
\theta_2 &= \frac{N_1}{N_2} \\
\theta_1 &= \frac{300}{50} = 6 \text{ rad} \\
\theta_3 &= \frac{N_3}{N_2} = \frac{50}{150} = \frac{1}{3} \text{ rad} \\
\theta_4 &= \frac{300}{150} = \frac{2}{3} \text{ rad} \\
\theta_5 &= \frac{300}{150} = \frac{2}{3} \text{ rad} \\
\theta_6 &= \frac{300 \times 15}{30} = 150 \text{ rad} \\
\omega_1 &= \frac{10}{T_2} = \frac{300}{50} = 6 \text{ rad/sec} \\
\omega_4 &= \frac{10}{T_3} = \frac{300}{150} = 2 \text{ rad/sec} \\
\ddot{\theta}_3 &= \frac{8 \text{ rad/sec}^2}{A_n} 
\end{align*}
\]

Example 1.21. Draw the \( f-V \) and \( f-i \) analogy of the mech. system shown in figure 1.53.

Solution: Corresponding the displacements \( x_1, x_2 \), \( x_3 \) and \( x_4 \) there are three meshes. The first mesh containing a voltage source \( V(F) \) and Inductance \( L_1(M_1) \), \( C_1(1/K_1) \) will be common between two meshes. Similarly \( C_2(1/K_2) \) and \( R_1(B_1) \) will be common between second and third meshes. The \( f-V \) & \( f-i \) analogous circuit are shown in fig. 1.54 and 1.55 respectively.

\( f-V \) analogy:

\[ \text{Fig. 1.54} \]

\( f-i \) analogy:

\[ \text{Fig. 1.55} \]

Example 1.22. Draw the electrical analogous circuit of the given mechanical system shown in fig. 1.56. Use \( f-V \) analogy.
Example 1.23. If \( K \) is the stiffness of the spring, draw the analogous electrical circuit based on \( f-V \) analogy and determine the transfer function in each case.

Solution: The electrical analogous circuit is shown in fig. 1.59.

From fig. 1.59:

\[
V_i = RI + \frac{1}{C} \int i \, dt \quad \text{(1.83)}
\]

\[
V_0 = \frac{1}{C} \int i \, dt \quad \text{(1.84)}
\]

Laplace transform of 1.83 & 1.84:

\[
V_i(s) = RI(s) + \frac{1}{C} \frac{1}{s} I(s) = \left(1 + \frac{RI}{C} \right) I(s)
\]

\[
V_0(s) = \frac{1}{C} \frac{1}{s} I(s) = \frac{1}{sC} I(s)
\]

\[
V_i(s) = \frac{1}{CS} I(s)
\]

\[
V_0(s) = \frac{1}{sC} I(s) = \frac{1}{s} \frac{1}{sC} I(s) = \frac{1}{s + RCS} I(s)
\]

Ans.

Example 1.24. Draw the electrical analogous circuit (use \( f-V \) analogy) and derive their transfer functions.

From fig. 1.58:

\[
\begin{align*}
B \dot{x}_2 &= K (x_1 - x_2) \\
\text{Laplace transform of eq. 1.86:} \\
SBX_2(s) &= KX_1(s) - KX_2(s) \\
X_2(s) = \frac{SB + K}{K} &= KX_1(s) \\
\therefore \frac{X_2(s)}{X_1(s)} &= \frac{K}{K + SB} \quad \text{Ans.} \quad \text{(1.87)}
\end{align*}
\]

Solution: Electrical analogous circuit is shown in fig. 1.61.

From fig. 1.61:

\[
V_i = \frac{1}{C_1} \int i \, dt + \frac{1}{C_2} \int i \, dt + Ri \quad \text{(1.88)}
\]

\[
V_0 = Ri + \frac{1}{C_2} \int i \, dt \quad \text{(1.89)}
\]

Laplace transform of 1.88 & 1.89:

\[
V_i(s) = \frac{1}{C_1 S} I(s) + \frac{1}{C_2 S} I(s) + RI(s) \quad \text{(1.90)}
\]

\[
V_0(s) = RI(s) + \frac{1}{C_2 S} I(s)
\]

\[
V_0(s) = \frac{1}{C_1 + C_2 + C_2 R S}
\]

from fig. 1.60:

\[
K_1 x_2 = B \frac{d}{dt} (x_1 - x_2) + K_2 (x_1 - x_2) \quad \text{(1.92)}
\]

Laplace transform of 1.92:

\[
K_1 X_2(s) = \frac{SB}{K_1} + K_2 X_1(s) \quad \text{(1.93)}
\]

\[
\begin{align*}
X_2(s) &= \frac{SB + K_2}{K_1 + K_2 + SB} \quad \text{(1.93)}
\end{align*}
\]
Example 1.25. Draw the electrical analogous circuit, use f-V and f-I analogy.

![Fig. 1.62](image1.png)

Fig. 1.62.

Solution: Required circuit shown in fig. 1.63 & 1.64.

![Fig. 1.63. F-V Analogous Circuits](image2.png)

![Fig. 1.64. F-I Analogous Circuits](image3.png)

Example 1.26. Show that the system shown in fig. 1.65(a), 1.65(b) are analogous system.

(RML Univ, Faizabad, 2002)

![Fig. 1.65.](image4.png)

Solution: From fig. 1.65a:

\[ B_2 \frac{d}{dt}(x_1 - x_0) + K_1(x_1 - x_0) = B_1 \frac{d}{dt}(x_0 - y) \]  
\[ B_2 \frac{d}{dt}(x_0 - y) = K_2 y \]  

Laplace transform of 1.94 & 1.95:

\[ B_1 SX_1(s) - B_2 SX_0(s) + K_1 X_1(s) - K_1 X_0(s) = SB_1 X_1(s) - SB_2 X_0(s) \]

\[ SB_2 X_0(s) - SB_1 X_0(s) = K_2 y(s) \]

\[ y(s) = \left( \frac{S K_1 + S B_1}{S K_2 + S B_2} \right) X_1(s) \]  

(1.98)

(from 1.97) find y(s) and put in 1.96.

using f-V analogy (from table 1.3) the eqn (1.98) becomes:

\[ c_0(t) = \frac{1 + R_1 C_S}{1 + R_1 C_S} \]  
\[ c_1(t) = \frac{1 + R_1 C_S}{1 + R_1 C_S} \]  

(1.99)

from fig (1.65b):

\[ Z_1 = \frac{R_1}{1 + R_1 C_S} \]  
\[ Z_2 = \frac{R_1}{1 + R_1 C_S} \]  

(1.100)

Since eqn (1.99) & eqn (1.100) are same, hence the given systems are analogous.

\[ B_2 \frac{d}{dt}(x_0 - y) = K_2 y \]  

(1.95)

Laplace transform of 1.94 & 1.95:

\[ B_1 SX_1(s) - B_2 SX_0(s) + K_1 X_1(s) - K_1 X_0(s) = SB_1 X_1(s) - SB_2 X_0(s) \]

\[ SB_2 X_0(s) - SB_1 X_0(s) = K_2 y(s) \]

from 1.97 find y(s) and put in in 1.96.

\[ B_1 SX_1(s) - B_2 SX_0(s) + K_1 X_1(s) - K_1 X_0(s) = SB_1 X_1(s) - SB_2 X_0(s) \]

\[ SB_2 X_0(s) - SB_1 X_0(s) = K_2 y(s) \]

(1.98)

using f-V analogy (from table 1.3) the eqn (1.98) becomes:

\[ c_0(t) = \frac{1 + R_1 C_S}{1 + R_1 C_S} \]  
\[ c_1(t) = \frac{1 + R_1 C_S}{1 + R_1 C_S} \]  

(1.99)

from fig (1.65b):

\[ Z_1 = \frac{R_1}{1 + R_1 C_S} \]  
\[ Z_2 = \frac{R_1}{1 + R_1 C_S} \]  

(1.100)
1.18. BLOCK DIAGRAM REPRESENTATION

Any system can be described by a set of differential equations or can be represented by the schematic diagram containing all components and their connections. But for complicated systems these two methods are not suitable. The block diagram representation is the combination of above two methods. A block may represent a single component or a group of components, but each block is completely characterized by a transfer function. The transfer function is an expression which relates output to input in s-domain. Transfer function does not give any information about the internal structure of the system. Once we determine the transfer function, then we can represent the system by the block diagram. Block diagrams are single line diagrams, that is the flow of system variables from one block to another block is represented by a single line. Fig. 1.66 shows the block diagram representation of a system.

\[ \frac{C(s)}{R(s)} = \frac{G(s)}{1 + G(s)} \]

Where,
- \( R(s) \) = Input
- \( C(s) \) = output
- \( G(s) \) = transfer function

Then, the system can be represented as
\[ C(s) = R(s), G(s) \]  

The flow of system variables from one block to another block is represented by the arrow. In addition to this, the sum of the signals or the difference of the signals are represented by a summing point (fig. 1.67a). Application of one input source to two or more blocks is represented by a take off point (fig. 1.67b).

1.19. HOW TO DRAW THE BLOCK DIAGRAM

Consider a simple R-L circuit shown in fig. 1.68
Apply KVL
\[ V_i = R_i + L \frac{di}{dt} \]  

The output of the summing point is given to block and output of the block is \( V_i \) as per eqn. 1.109.

\[ V_i(s) = V_i(s) - V_d(s) \]

Form eqn. 1.110 the output of block \( V_i(s) \) is given to another block containing the element SL & the output of the second block is \( V_o \).

Combining the fig. 1.70 & 1.71 we get required block dia.

Example 1.27. Draw the block diagram of series RLC circuit, where \( V_i \) & \( V_o \) are the input and output voltages.

Solution: The transformed network in S-domain is shown in fig. 1.74.
Example 1.26. Draw the block diagram of the circuit shown in fig. 1.79.

Solution:

\[ i_1(t) = \frac{V_1(t) - V_2(t)}{R_1} \quad ... (1.113a) \]

\[ V_2(t) = \frac{1}{C_2} \int [i_1(t) - i_2(t)] dt \quad ... (1.113b) \]

\[ i_2(t) = \frac{1}{R_2} [V_3(t) - V_0(t)] \quad ... (1.113c) \]

\[ V_0(t) = \frac{1}{C_3} \int i_2(t) dt \quad ... (1.113d) \]
Solution: From fig. 1.81

\[
\begin{align*}
\dot{V} &= V_1 - V_o \\
V_1 - V_o &= \frac{\dot{V}}{R_2} \\
V_o &= L \frac{\dot{V}}{dt} \\
\dot{L} &= \frac{1}{SL} V_o \\
\end{align*}
\]

Laplace transform of above eqns

\[
\begin{align*}
\dot{V}(s) &= \frac{1}{SL} [V_1(s) - V_o(s)] \\
V_1(s) &= R_2 [I(s) - L(s)] \\
V_o(s) &= SL \dot{L}(s) \\
L(s) &= \frac{1}{SL} V_o(s)
\end{align*}
\]

From 1.114d, 1.114e, and 1.114f, we can draw the block diag.

1.20. CLOSED LOOP CONTROL SYSTEM

A closed loop system is one in which output is fed back into an error detector and compared with the reference input. The feedback may be negative or positive.

Consider a closed loop system shown in fig. 1.83

\[
\begin{align*}
R(s) & \xrightarrow{G(s)} E(s) \\
E(s) & \xrightarrow{H(s)} C(s)
\end{align*}
\]

where,

- \( R(s) \) = Reference input
- \( E(s) \) = Actuating signal or error signal
- \( G(s) \) = Forward path transfer function
- \( C(s) \) = Output signal
- \( H(s) \) = Feedback transfer function
- \( B(s) \) = Feedback signal

From fig. 1.83

\[
\begin{align*}
C(s) &= C(s) \cdot E(s) \\
B(s) &= H(s) \cdot C(s) \\
E(s) &= R(s) - B(s)
\end{align*}
\]

Put the value of \( C(s) \) from 1.115a in 1.115b

\[
\begin{align*}
B(s) &= H(s) \cdot G(s) \cdot E(s) \\
B(s) &= H(s) \cdot G(s) \cdot H(s)
\end{align*}
\]

\[
E(s) = \frac{G(s)}{1 + G(s)H(s)}
\]

Input-Output Relationship / 35.
1.21. MULTINPUT-MULTOUTPUT SYSTEM (MIMO)

When two or more inputs act on a system, each input can be treated independently of the others. Complete output can be obtained by adding the effect of each input.

Consider a two-input linear system shown in fig. 1.85.

```
\[
\frac{C(s)}{R_1(s)} = \frac{G_1(s)}{1 + G_1(s)G_2(s)} \quad \text{For negative feedback}
\]
\[
\frac{C(s)}{R_2(s)} = \frac{G_2(s)}{1 - G_1(s)G_2(s)} \quad \text{For positive feedback}
\]
```

**Step 1:** Put \( R_1(s) = 0 \)

\[
\frac{C_2(s)}{R_2(s)} = \frac{G_2(s)}{1 + G_1(s)G_2(s)H(s)}
\]

**Step 2:** Put \( R_2(s) = 0 \)

\[
\frac{C_1(s)}{R_1(s)} = \frac{G_1(s)G_2(s)}{1 + G_1(s)G_2(s)H(s)}
\]

Hence,

\[
C(s) = C_1(s) + C_2(s)
\]

\[
C(s) = \frac{G_1(s)G_2(s)R_1(s)}{1 + G_1(s)G_2(s)H(s)} + \frac{G_2(s)R_2(s)}{1 + G_1(s)G_2(s)H(s)}
\]

\[
C(s) = \frac{G_1(s)G_2(s)}{1 + G_1(s)G_2(s)H(s)} [R_1(s)G_1(s) + R_2(s)]
\]

MIMO system can be expressed in matrix form

\[
\begin{bmatrix}
C_1(s) \\
C_2(s) \\
\vdots \\
C_m(s)
\end{bmatrix} = \begin{bmatrix}
G_1(s)G_2(s) & \cdots & G_1(s)G_m(s) \\
\vdots & \cdots & \vdots \\
G_2(s) & \cdots & G_m(s)
\end{bmatrix} \begin{bmatrix}
R_1(s) \\
R_2(s) \\
\vdots \\
R_m(s)
\end{bmatrix}
\]

where,

\[
C(s) = G(s)R(s)
\]

\[
G(s) = \text{matrix transfer function}
\]

1.22. BLOCK DIAGRAM REDUCTION

When a number of blocks are connected, the overall transfer function can be obtained by block diagram reduction technique. The following rules are associated with the block reduction technique.

**Rule No. 1. Blocks in cascade**

When two or more blocks are in cascade, the resultant block is a product of the individual block transfer function. Consider the two blocks are in cascade shown in fig. 1.86.

```
\[
\frac{C(s)}{R(s)} = \frac{G_1(s)G_2(s)}{1 + G_1(s)G_2(s)}
\]
```

**Rule No. 2. Blocks in Parallel**

When two or more blocks are connected in parallel as shown in fig. 1.88a, the resultant block is the sum of individual block transfer function.

```
\[
\frac{C(s)}{R(s)} = \frac{G_1(s)}{1 + G_1(s)G_2(s)H(s)} + \frac{G_2(s)}{1 + G_1(s)G_2(s)H(s)}
\]
```

from fig. 1.88a

\[
C(s) = R(s)G_1(s) + R(s)G_2(s) + R(s)G_3(s)
\]

\[
= R(s)[G_1(s) + G_2(s) + G_3(s)]
\]

The equivalent diag. is shown in fig. 1.88b

**Rule No. 3. Moving a take off point ahead of a block**

If a take off point is moved ahead of a block, a block with same transfer function will introduce in the branch of a take off point, as shown in fig. 1.88c.

```
\[
\frac{C(s)}{R(s)} = \frac{G_1(s)G_2(s)}{1 + G_1(s)G_2(s)H(s)}
\]
```

**Rule No. 4. Moving a take off point after the block**

If a take off point is moved after the block, a block with the reciprocal of the transfer function is introduced in the branch of a take off point as shown in fig. 1.88d.

```
\[
\frac{C(s)}{R(s)} = \frac{G_1(s)G_2(s)}{1 + G_1(s)G_2(s)H(s)}
\]
```

**Rule No. 5. Moving a summing point beyond a block**
Example 1.30. Derive the transfer function using block reduction technique.

Solution: Step 1: Shift the pick off point beyond the block $1/sC_2$

$$\frac{R(s)}{V(s)} = \frac{1}{1 + sC_2}$$

Step 2: Two blocks are in cascade

Step 3:

Example 1.31. Find the overall transfer function of the system shown in fig. 1.80c.

Solution: Step 1: Shift the pick off point beyond the block $1/sC_2$

Step 2: Two blocks are in cascade

Step 3:

Solution: Step 1: There are two internal closed loops, remove these loops by using the eqn 1.115f.

Step 2: Two blocks are in cascade, use the rule no. 1.

Step 3: Fig 1.89b is a closed loop, again use eqn 1.115f.
Example 1.32. Determine the ratio $C(s)/R(s)$ for the system shown in fig. 1.91.

$$\begin{align*}
\frac{V(s)}{V_i(s)} &= \frac{1}{1 + R_2 C_2 s^2 + (R_1 C_1 + R_2 C_2 + R_4 C_2) s} \\
&\text{Ans.}
\end{align*}$$

Example 1.33. Determine the ratio $C/R$. 

Solution: Step 1: Shift the takeoff point beyond the block $C_5$. 

Step 2: $C_2$ & $C_3$ are in cascade.

Step 3:

Step 4:

Step 5:

Step 6:

$$\begin{align*}
\frac{C(s)}{R(s)} &= \frac{G_i G_3 G_5}{1 + G_i G_3 H_2 + G_i G_3 H_1 + G_i G_3 G_5} \\
&\text{Ans.}
\end{align*}$$
Solution:

\[
E_2 = R - B_2 \\
B_2 = X_1 G_1 \\
X_2 = R - B_2 \\
X_2 = G_2 (R - B_2) \\
B_3 = X_2 G_4 \\
B_3 = G_4 G_2 (R - X_2 G_3) \\
E_3 = R - B_3 = R - G_3 G_4 (R - X_3 G_3) \\
X_3 = E_3 G_1 \\
X_3 = G_1 [R - G_3 G_4 G_3 (R - X_3 G_3)] \\
X_3 = G_1 R - G_3 G_4 G_3 R + G_3 G_2 G_3 G_4 X_1 \\
X_1 (1 - G_3 G_2 G_3 G_4) = R (G_1 - G_3 G_4 G_3 G_4) \\
R = \frac{1}{1 - G_3 G_2 G_3 G_4} \\
\]

from eqn (1.118b) the value of \(x_1 \) in eqn (1.118c)

\[
X_2 = G_3 \left[ R - G_2 \frac{(1 - G_2 G_3 G_4 R)}{1 - G_2 G_3 G_4} \right] = G_3 R \frac{(1 - G_2 G_3)}{1 - G_2 G_3 G_4 G_4} \\
\]

Example 1.34. Determine the transfer function \(C/R \), \(C/R_2 \), \(C/R_1 \), and \(C/R_2 \), from the block diagram shown in fig. 1.93.


Solution:

\[
E_2 = R - B_2 \\
B_2 = C_1 H_2 \\
C_2 = E_2 G_3 \\
\]

put the value of \(E_2 \) from 1.119a in 1.119c

\[
C_2 = G_2 (R_2 - B_2) \\
\]

from (1.119b) the value of \(B_2 \) in 1.119d

\[
C_2 = G_2 (R_2 - C_2 H_2) \\
B_1 = C_2 H_1 \\
\]

put the value of \(C_2 \) from 1.119c in 1.119f

\[
B_1 = C_1 H_1 (R_2 - C_2 H_2) \\
E_1 = R_2 - B_1 \\
E_1 = R_1 - G_1 H_2 (R_2 - C_2 H_2) \\
C_1 = E_1 G_1 \\
\]

Also,

from (1.119b) the value of \(E_1 \) in 1.119g

\[
C_1 = G_1 [R_1 - G_1 H_1 (R_2 - C_2 H_2)] \\
C_1 (1 - G_1 G_2 H_1 H_2) = G_1 R_1 - G_2 G_1 H_1 H_2 R_2 \\
\]

When \(R_1 = 0 \) from (1.119h)

\[
C_1 = \frac{G_1}{1 - G_1 G_2 H_1 H_2} \\
\]

from (1.119i) the value of \(C_1 \) in 1.119i

\[
C_2 = \frac{G_2}{R_2 + \frac{G_2 G_3 H_1 H_2 R_2}{1 - G_2 G_3 G_4 G_4}} \\
\]

When \(R_2 = 0 \) from (1.119j) becomes

\[
C_2 = \frac{G_2}{1 - G_2 G_3 G_4 G_4} \\
\]

When \(R_2 = 0 \) from (1.119k) becomes

\[
C_2 = \frac{-G_2 G_3 H_1 H_2}{1 - G_2 G_3 G_4 G_4} \\
\]

from (1.119n) the value of \(C_1 \) in 1.119n

\[
C_2 = \frac{G_2}{R_2 + \frac{G_2 G_3 H_1 H_2 R_2}{1 - G_2 G_3 G_4 G_4}} \\
\]

When \(R_2 = 0 \) from (1.119o) becomes

\[
C_2 = \frac{G_2}{1 - G_2 G_3 G_4 G_4} \\
\]

When \(R_2 = 0 \) from (1.119p) becomes

\[
C_2 = \frac{-G_2 G_3 H_1 H_2}{1 - G_2 G_3 G_4 G_4} \\
\]
Example 1.35. Determine the ratio $C(s)/R(s)$.

Solution: Step 1: Shift the takeoff point beyond block $G$.

Step 2:

Step 3:

Example 1.36. Find the ratio $C(s)/R(s)$ of the system shown in fig. 1.95.

Solution: Step 1: Shift the takeoff point beyond block $G_3$.

Step 2:

Example 1.37 Determine the transfer matrix MIMO system of example 1.34.

Solution:

Example 1.38. Find the transfer function of the system shown in fig. 1.96.
Solution: Step 1:

Apply KVL in armature circuit

\[ V = R_i I_a + L \frac{dI_a}{dt} + E \]  
\[-(1.129)\]

Since, field current \( I_f \) is constant, the flux \( \Phi \) will be constant

When armature is rotating, an e.m.f is induced

\[ E \propto \Phi \wedge \omega \]

\[ E = K_b \Phi \frac{d\theta}{dt} \]  
\[-(1.121)\]

or,

\[ E = K_b \Phi \frac{d\theta}{dt} \]

Where,

\( \omega \) = angular velocity

\( K_b \) = back e.m.f constant

Now, the torque \( T \) delivered by the motor will be the product of armature current and flux

\[ T = K I_a \Phi \]  
\[-(1.122)\]

where \( K \) = motor torque constant

The dynamic equation with moment of inertia & coeff. of friction will be

\[ T = \int \frac{d^2 \theta}{dt^2} + B \frac{d\theta}{dt} \]  
\[-(1.123)\]

Take the laplace transform of equations 1.120, 1.121, 1.122 & 1.123

\[ V(s) - E(s) = I_a(s) (R_r + sL_a) \]

\[ E(s) = K_b \Phi 50(s) \]

\[ T(s) = K I_a(s) \]

\[ T(s) = (s^2 + \alpha) \Phi(s) \]

\[ T(s) = (s^2 + B) 50(s) \]

The block diag. for each equation

Combine all four block diagrams

Fig. 1.97: Block diagram of armature controlled d.c motor
Now determine the transfer function by block reduction method.

\[ \frac{\theta(s)}{V(s)} = \frac{K}{s(\tau_a + s\tau_m)(\tau_a + s)(\tau_f + s)} \]

Equation 1.124 can be written as

\[ \frac{\theta(s)}{V(s)} = \frac{K}{R_s(1 + s\tau_m)(1 + s\tau_f)} \]

Put \( \frac{L}{R_s} = \tau_a \) time constant of armature circuit

\[ \frac{1}{\tau_f} = \tau_m = \text{mechanical time constant} \]

equation 1.124 becomes

\[ \frac{\theta(s)}{V(s)} = \frac{K}{s(\tau_a + s\tau_m)(1 + s\tau_f)} \]

From the block diag. 1.98 it is clear that it is a closed loop system. The effect of the back e.m.f is represented by the feedback signal proportional to the speed of the motor.

1.23.2. Field Control d.c. Motor

\[ V_f = R_f I_f + L_f \frac{d}{dt} I_f \]  
\[ \theta = K V_f \]

1. A Constant current \( I_f \) is fed to the armature.
2. Flux is proportional to the field current. \( \phi \propto I_f \)
3. Apply KVL in field circuit

\[ V_f = R_f I_f + L_f \frac{d}{dt} I_f \]  
\[ \theta = K V_f \]

Dynamic equation of torque in terms of \( I \) & \( B \)

\[ T = \frac{d}{dt} \theta + B \frac{d}{dt} \theta \quad \text{(1.128)} \]

Laplace transformation of eqn (1.127), (1.128) & (1.129)

\[ V_f(s) = R_f I_f(s) + sL_f I_f(s) \]

Put the value of \( I_f(s) \) from (1.130) in (1.131)

\[ T(s) = K \frac{V_f(s)}{R_f + sL_f} \quad \text{(1.133)} \]

From equation (1.132) & (1.133)

\[ \theta(s) [s^2 + sB] = K \frac{V_f(s)}{R_f + sL_f} \]

Equation 1.134 can be written as

\[ V_f(s) = \frac{KK_f}{R_f \left(1 + \frac{s}{\tau_f} \left(1 + \frac{s}{\tau_m}\right)\right)} \]

where

\[ \tau_m = \frac{1}{\tau_f} \quad \text{mechanical time constant} \]

\[ \tau_f = \frac{L_f}{R_f} \quad \text{time constant for field circuit} \]
1.24. SIGNAL FLOW GRAPH

The process of block diagram reduction technique is time consuming because at every stage modified block diagram is to be redrawn. A simple method was developed by S.J. Mason which is known as signal flow graph. This method is very simple and does not require any reduction technique. Signal flow graph is applicable to the linear systems.

A signal flow graph is a diagram which represents a set of simultaneous equations. Signal flow graph consists of nodes and these nodes are connected by a directed line called branches. Every branch of signal flow graph having an arrow, which represents the flow of signal.

The following terms are associated with the signal flow graph.

Fig. 1.101

1. Input node or source node: An input node is a node which has only outgoing branches. e.g., $x_1$ is the input node.
2. Output node or sink node: An output node is a node that has only one or more incoming branches. e.g., $x_5$ is the output node.
3. Mixed nodes: A node having incoming and outgoing branches is known as mixed nodes. e.g., $x_2$, $x_3$, and $x_4$ are the mixed nodes.
4. Transmittance: Transmittance also known as transfer function, which is normally written on the branch near the arrow e.g., $H_{12}$, $H_{23}$ etc.
5. Forward path: Forward path is a path which originates from the input node and terminates at the output node and along which no node is traversed more than once. e.g., in fig 1.101 there are two forward paths.
   1. $x_1$ to $x_3$ to $x_4$ to $x_5$ to $x_6$
   2. $x_1$ to $x_2$ to $x_4$ to $x_5$ to $x_6$
6. Loop: Loop is a path that originates and terminates on the same node and along which no other node is traversed more than once. e.g., $x_3$ to $x_5$ to $x_2$ to $x_4$ to $x_3$.
7. Self loop: It is a path which originates and terminates on the same node. e.g., $x_4$ to $x_4$.
8. Path gain: The product of the branch gains along the path is called path gain. e.g., The gain of the path $x_1$ to $x_2$ to $x_4$ to $x_5$ to $x_6$ is $a_{12}a_{23}a_{34}a_{45}a_{56}$.
9. Loop gain: The gain of the loop is known as loop gain. e.g., the gain of the loop $x_2$ to $x_3$ to $x_4$ is $a_{23}a_{34}$.
10. Non-touching loops: Non touching loops having no common nodes branch and paths. e.g., The loops $x_3$ to $x_2$ to $x_4$ and $x_4$ to $x_3$ are non touching loops.

1.25. CONSTRUCTION OF SIGNAL FLOW GRAPH FROM EQUATIONS

Consider the following sets of equations

\[
\begin{align*}
    y_2 &= f_{11} y_1 + f_{23} y_3 \\
    y_3 &= f_{32} y_2 + f_{33} y_3 + f_{31} y_1 \\
    y_4 &= f_{43} y_3 + f_{42} y_2 \\
    y_5 &= f_{54} y_4 \\
    y_6 &= f_{64} y_5 + f_{64} y_4
\end{align*}
\]

where $y_1$ is the input and $y_6$ is the output.

First of all draw the nodes. In the given example there are six nodes. From the first equation it is clear that the $y_2$ is the sum of two signals. Similarly, $y_3$ is the sum of three signals & so on. Insert the branches with proper transmittance to connect the nodes.

**Step 1:** Draw the nodes

**Step 2:** Draw the SFG for eqn (1)
1.26. SIGNAL FLOW GRAPH FOR DIFFERENTIAL EQUATIONS

Consider the following differential equation
\[ y'' + 3y' + 5y + 2y = x \]  

**Step 1:** Solve the eqn 1.135 for the highest order
\[ y'' = x - 3y' - 5y - 2y \]  

**Step 2:** Consider the left hand term (highest order derivative) as dependent variable and all other terms on right hand side as independent variables.

Construct the branches of signal flow graph as shown in fig. (1.103a).

**Step 3:** Connect the nodes of highest order derivative to the node whose order is lower than this and so on. The flow of the signal will be from higher node to the lower order node and transmittance will be \( \frac{1}{s} \), as shown in fig. 1.103b.

**Step 4:** Reverse the sign of a branch connecting the \( q \)th node to the \( q+1 \)th node of a signal flow graph without disturbing the transfer function.

Consider the fig. 1.103b, reverse the sign of the branch connecting \( y'' \) to \( y' \), it is necessary to reverse the sign of all remaining branches entering as well as leaving the \( q \)th node.

Similarly, reverse the sign of branch connecting \( y' \) to \( y' \).

By reversing the sign, we have already reverse the sign of branch connecting \( y' \) to \( y \) and therefore further reversal of sign is not required.

**Step 5:** Redraw the signal flow graph (SFG)

1.27. CONSTRUCTION OF SIGNAL FLOW GRAPH FROM BLOCK DIAGRAM

**Rules:**
1. All variables, summing points and take off points are represented by nodes.
2. If a summing point is placed before a takeoff point in the direction of signal flow, in such case represent the summing point and takeoff point by a single node.
3. If a summing point is placed after a takeoff point in the direction of signal flow, in such case represent the summing point and takeoff point by separate nodes connected by a branch having transmittance unity.

Consider the block diagram shown in Fig 1.104(a), the corresponding SFG is shown in Fig 1.104(b).

1.28. MASON'S GAIN FORMULA
The overall transmittance or graph transmittance between the source node and sink node is given by Mason's gain formula. Mason's gain formula for signal flow graph is as follows:

\[ T = \frac{\sum s_i \Delta_i}{\Delta} \]

where, \( T \) = transfer function
\( \Delta = 1 - \) [sum of all individual loop gain] + [sum of all possible gain products of two non-touching loops] - [sum of all possible gain products of three non-touching loops] + ... 
\( s_i \) = gain of the \( k \)th forward path
\( \Delta_i \) = the part of \( \Delta \) not touching the \( k \)th forward path.

Consider the signal flow graph shown in Fig 1.104(b). There are two forward paths ((i) 1-2-3-4 (ii) 1-4). Therefore gain of the two paths will be:
\( s_1 = G_2G_3 \)
\( s_2 = G_4 \)

There are three individual loops:
\( L_1 = G_2G_3 \) \hspace{1cm} (2-3-2)
\( L_2 = -G_3G_5 \) \hspace{1cm} (1-2-3-4-1)
\( L_3 = -G_5G_2 \) \hspace{1cm} (1-4-1)

Since all three loops touching the forward path \( g_1 \), therefore \( \Delta_1 = 1 - 0 = 1 \). The first loop \( L_1 \) does not touch the forward path \( g_2 \), therefore \( \Delta_2 = 1 - G_2G_3 \).
There are two non-touching loops \( L_1 \) & \( L_2 \):
\( L_1L_2 = -G_3G_5 \)

Apply Mason's gain formula:
\[ T = \frac{C}{R} = \frac{\sum s_i \Delta_i}{\Delta} \]
\[ T = \frac{G_2G_3 + G_4(1 - G_2G_3)}{1 - (G_2G_3 + G_3G_5 - G_5G_2 + G_5G_2G_3G_5)} \]
\[ \frac{C}{R} = \frac{G_2G_3 + G_4 - G_4G_5}{1 - G_2G_3 - G_3G_5 + G_4G_5 + G_5G_2 - G_5G_2G_3G_5} \]

Note: where \( \Delta = 1 - (L_1 + L_2 + L_3) + (L_4L_2) \)

Example 1.39. Draw the signal flow graph for the following set of equations.
\[ x_2 = x_1 + ax_5 \]
\[ x_3 = bx_3 + cx_4 \]
\[ x_4 = dx_2 + ex_3 \]
\[ x_5 = fx_4 + gx_3 \]
\[ y_5 = x_5 \]

Solution: The required signal flow graph is:

Example 1.40. For the system represented by the given equations find the transfer function \( x_4/x_1 \) by the help of signal flow graph technique.
\[ x_2 = a_1x_1 + a_2x_2 + a_3x_4 + a_5x_5 \]
\[ x_3 = a_3x_3 \]
\[ x_4 = a_3x_5 + a_4x_4 \]
\[ x_5 = a_5x_3 + a_6x_4 \]

where \( x_1 \) is the input variable and \( x_5 \) is the output variable.

Solution: There are two forward paths (1) \( x_1 \) to \( x_2 \) to \( x_3 \) to \( x_4 \) to \( x_5 \)
(2) \( x_1 \) to \( x_2 \) to \( x_3 \) to \( x_4 \)

(R.M.L. University Faisalabad, 2002)
The gain of the paths

\[ g_1 = a_{12} a_{23} a_{34} a_{45} \]
\[ g_2 = a_{12} a_{23} a_{35} \]

Gain of individual loops

\[ L_1 = a_{23} a_{32} \]
\[ L_2 = a_{23} a_{34} a_{42} \]
\[ L_3 = a_{23} a_{34} a_{45} a_{52} \]
\[ L_4 = a_{23} a_{35} a_{52} \]
\[ L_5 = a_{34} a_{44} \]

Gain of two non-touching loops

\[ L_1 L_2 = a_{23} a_{32} a_{44} \]
\[ L_1 L_3 = a_{23} a_{35} a_{52} a_{44} \]

Since all the loops touch the forward path \( L_1 \) : \( \Delta_1 = 1 - 0 = 1 \)
loop \( L_5 \) do not touch the second forward path : \( \Delta_2 = 1 - a_{44} \)

\[ \frac{x_2}{x_1} = \frac{a_{12} a_{23} a_{32} + a_{12} a_{23} a_{34} a_{42}}{1 - (a_{23} a_{34} a_{42} + a_{23} a_{35} a_{52} a_{44})} \]

\[ \frac{x_2}{x_1} = \frac{a_{12} a_{23} a_{34} a_{42} + a_{12} a_{23} a_{35} a_{52} a_{44}}{1 - (a_{23} a_{34} a_{42} + a_{23} a_{35} a_{52} a_{44})} \]

Example 1.41. For the given signal flow graph find the ratio \( C/R \).

Example 1.42. Obtain signal flow graph representation for a system whose block diagram is given below and using Mason's gain formula determine the ratio \( C/R \).

(RML Univ. Faisalabad 2001 Linear system theory)
Example 1.43. Obtain the transfer function for \( \frac{C(s)}{R(s)} \) given block diagram using block diagram reduction technique or signal flow graph (Mason's gain formula).

\[
\text{Solution: By signal flow graph SFG is shown in fig 1.109a.}
\]

The gain of the forward paths:
\[
\begin{align*}
\delta_1 &= G_1 G_2 G_3 \\
\delta_2 &= G_1 \\
\delta_3 &= 1
\end{align*}
\]

Individual loops:
\[
\begin{align*}
L_1 &= -G_1 G_2 H_1 \\
L_2 &= -G_1 G_3 H_1 \\
L_3 &= -G_1 G_4 H_1 \\
L_4 &= G_1 G_2 G_3 H_1 \\
L_5 &= G_1 G_2 G_3 H_2 \\
L_6 &= G_1 G_2 G_3 H_3 \\
L_7 &= G_1 G_2 G_3 H_4 \\
L_8 &= G_1 G_2 G_3 H_5 \\
L_9 &= G_1 G_2 G_3 H_6 \\
L_{10} &= G_1 G_2 G_3 H_7 \\
L_{11} &= G_1 G_2 G_3 H_8 \\
L_{12} &= G_1 G_2 G_3 H_9
\end{align*}
\]

Example 1.44. Draw the signal flow graph and determine \( \frac{C}{R} \) for the block dia shown in fig 1.110.

Solution: The SFG is shown in fig 1.110a.

\[
\begin{align*}
\delta_1 &= 1 + G_1 G_2 - G_2 H_3 \\
\delta_2 &= 1 - G_1 G_2 \\
\delta_3 &= G_1 G_2 \\
\delta_4 &= 1 + G_1 G_2 - G_2 H_4 \\
\delta_5 &= 1 + G_1 G_2 - G_2 H_5 \\
\delta_6 &= 1 + G_1 G_2 - G_2 H_6 \\
\delta_7 &= 1 + G_1 G_2 - G_2 H_7 \\
\delta_8 &= 1 + G_1 G_2 - G_2 H_8 \\
\delta_9 &= 1 + G_1 G_2 - G_2 H_9
\end{align*}
\]

Individual loops:
\[
\begin{align*}
L_1 &= -G_1 G_2 H_1 \\
L_2 &= -G_1 G_3 H_1 \\
L_3 &= -G_1 G_4 H_1 \\
L_4 &= G_1 G_2 G_3 H_1 \\
L_5 &= G_1 G_2 G_3 H_2 \\
L_6 &= G_1 G_2 G_3 H_3 \\
L_7 &= G_1 G_2 G_3 H_4 \\
L_8 &= G_1 G_2 G_3 H_5 \\
L_9 &= G_1 G_2 G_3 H_6 \\
L_{10} &= G_1 G_2 G_3 H_7 \\
L_{11} &= G_1 G_2 G_3 H_8 \\
L_{12} &= G_1 G_2 G_3 H_9
\end{align*}
\]

Two non-touching loops: None

\[
\begin{align*}
\frac{C}{R} &= \frac{8(\delta_1 + \delta_2)}{\Delta} \\
\frac{C}{R} &= \frac{G_1 G_2 G_3 G_4 + G_1 G_2 G_3 G_5 + G_1 G_2 G_3 G_6}{1 + G_1 G_2 - G_2 H_4}
\end{align*}
\]
Example 1.45. Obtain the transfer function $C/R$ of the block diagram shown in Fig. 1.111.

Solution:

\[ \frac{V_o}{R} = \frac{V_1}{R_1} - \frac{V_2}{R_2} \]

\[ V_2 = \frac{V_2}{R^2} \]

\[ i_2 = \frac{V_2}{R_2} \]

\[ V_3 = \frac{V_3}{R_3} \]

Fig. 1.111(b) Signal flow graph.

Example 1.47. Obtain the transfer function $C/R$ from the signal flow graph shown in Fig. 1.113.

Solution: From 1.112:

\[ i_1 = \frac{V_1}{R_1} - \frac{V_2}{R_2} \]

\[ V_2 = i_3 R_3 = \frac{(i_3 - i_2) R_3}{R_3} = \frac{R_3 i_1 - R_3 i_2}{R_3} \]

\[ i_2 = \frac{V_2}{R_2} \]

\[ V_3 = \frac{V_3}{R_3} \]

Fig. 1.113.

Solution: The gain of the forward paths $g_1 = G_2 G_4 G_6$

\[ g_2 = G_2 G_4 G_7 \]

\[ g_3 = G_2 G_4 G_7 \]

\[ g_4 = G_2 G_4 G_7 \]

\[ g_5 = G_2 G_4 G_7 \]

\[ g_6 = G_2 G_4 G_7 \]

Individual loops:

\[ L_1 = -G_4 H_1 \]

\[ L_2 = -G_4 H_2 \]

\[ L_3 = H_4 G_4 H_2 G_8 \]

Two non-touching loops:

\[ L_1 L_2 = G_4 H_1 G_2 H_2 \]

\[ \Delta_1 = 1 + G_4 H_2 \]

\[ \Delta_2 = 1 + G_4 H_1 \]

\[ \Delta_3 = 1 \]

\[ \Delta_4 = 1 \]

\[ \Delta_5 = 1 \]

\[ \Delta_6 = 1 \]

Example 1.46. Draw the signal flow graph for the network shown in Fig. 1.112, take $V_3$ as output node.

Fig. 1.112

Fig. 1.112(m)
1.29. BLOCK DIAGRAM FROM SIGNAL FLOW GRAPH

Step 1: For given signal flow graph, write the system equations.
Step 2: At each node consider the incoming branches only.
Step 3: Add all incoming signals algebraically at a node.
Step 4: For + or - sign in system equations use a summing point.
Step 5: For the gain of each branch of signal flow graph draw the block having the same transfer function as the gain of the branch.

Consider the following examples:

Example 1.49. Draw the block diagram from the given signal flow graph.

Solution: The system equations are:
- At node $x_1$, the incoming branches are from $R(s)$ and $x_2$
  $$x_1 = 1 - R(s) - x_2$$
- At node $x_2$, there are two incoming branches
  $$x_2 = x_1 - H_1 x_4$$
- At node $x_3$, there are two incoming branches
  $$x_3 = G_1 x_1 - H_2 x_5$$
Similarly at node $x_4$ and $x_5$, the system equations are:
- $x_4 = G_3 x_2$
- $x_5 = G_4 x_3 + G_5 x_3$

Draw the block diagram for each system equation.

For:
- $x_1 = R(s) - x_2$

Block diagram for:
- $x_2 = x_1 - H_1 x_4$
- $x_3 = G_1 x_1 - H_2 x_5$

Combining all above block diagrams:

1.30. EFFECT OF PARAMETER VARIATIONS

In control systems, the feedback reduces the error, also reduces the sensitivity of the system to parameter variations. The parameter may vary due to some change in conditions. The variation in parameter affects the performance of the system. So, it is necessary to make the system insensitive to such parameter variations.

1.30.1 Effect of Parameter Variations in Open Loop Control System

The open loop control system is shown in fig 1.116. From fig 1.116

$$\frac{C(s)}{R(s)} = G(s)$$

or,

$$C(s) = G(s) \cdot R(s)$$

Let

$$\Delta G(s)$$ Change in $G(s)$ due to parameter variations

$$\Delta C(s)$$ Corresponding change in output

From equation 1.137

$$C(s) + \Delta C(s) = [G(s) + \Delta G(s)] \cdot R(s)$$

Since,

$$G(s) \cdot R(s) = C(s)$$

$$C(s) + \Delta C(s) = C(s) + \Delta G(s) \cdot R(s)$$

or,

$$\Delta C(s) = \Delta G(s) \cdot R(s)$$
**1.30.2. Effect of Parameter Variations in Closed Loop System**

The closed loop system is shown in Fig. 1.117.

\[
\frac{C(s)}{R(s)} = \frac{G(s)}{1 + G(s)H(s)} \quad \text{(1.139)}
\]

or,

\[
C(s) = \frac{G(s)}{1 + G(s)H(s)} R(s)
\]

\[
C(s) + \Delta C(s) = \frac{G(s) + \Delta G(s)}{1 + G(s)H(s)} R(s)
\]

\[
= \frac{G(s) + \Delta G(s)}{1 + G(s)H(s)} \frac{1}{H(s)} R(s)
\]

Since, \( \Delta G(s) H(s) \ll [1 + G(s) H(s)] \), neglect \( \Delta G(s) H(s) \)

\[
C(s) + \Delta C(s) = \frac{G(s) + \Delta G(s)}{1 + G(s)H(s)} R(s)
\]

\[
\frac{G(s)}{1 + G(s)H(s)} R(s) + \frac{\Delta G(s)}{1 + G(s)H(s)} R(s)
\]

\[
C(s) = \frac{G(s)}{1 + G(s)H(s)} R(s)
\]

Equation 1.140 gives the change in output due to parameter variations in \( G(s) \) in a closed loop system.

Generally

\[
\frac{G(s) H(s)}{1 + G(s)H(s)} \gg 1
\]

from equation 1.140 it is clear that the change in output is reduced due to parameter variations in \( G(s) \) by \([1 + G(s) H(s)]\). But in open loop system there is no reduction because of no feedback.

**1.30.3 Effect of Feedback on Sensitivity**

The parameters of any control system changes with the change in environment conditions, the performance of the system. These parameter variations affects due to the change in temperature during its operation.

So, a control system should be insensitive to the parameter variations. Let \( P \) is a gain parameter, \( R \) is a parameter \( P \) to the parameter \( R \) is

\[
S_P^R = \frac{d}{R} \left( \ln R \right) = \frac{1}{R} \frac{\partial R}{\partial P} \frac{\partial P}{\partial R}
\]

In general 'R' may be the output variable and 'P' may be the gain, the feedback factor etc.

\[
T(s) = \text{Overall transfer function}
\]

\[
G(s) = \text{Forward path transfer function}
\]

Then, sensitivity will be

\[
S_C^T = \frac{\partial T(s)}{\partial G(s)} = \frac{T(s)}{G(s)} \quad \text{(1.141)}
\]

For open loop system

\[
T(s) = G(s)
\]

\[
S_C^T = \frac{\partial G(s)}{\partial G(s)} = 1
\]

Thus, the sensitivity of open loop system is unity.

Sensitivity of closed loop system:

\[
T(s) = \frac{G(s)}{1 + G(s)H(s)} \quad \text{(1.142)}
\]

\[
\frac{\partial T(s)}{\partial G(s)} = \frac{G(s)}{1 + G(s)H(s)} \left[ \frac{1}{1 + G(s)H(s)} \right] = \frac{1}{1 + G(s)H(s)}
\]

Sensitivity is given by from eqn. 1.141

\[
S_C^T = \frac{G(s)}{1 + G(s)H(s)} \quad \text{(1.143)}
\]

put the values of \( T(s) \) & \( \partial T(s) / \partial G(s) \)

\[
S_C^T = \frac{G(s)}{1 + G(s)H(s)} \quad \text{(1.143)}
\]

From equation (1.143) the sensitivity is reduced due to the feedback by a factor \( 1 / (1 + G(s)H(s)) \) as compared to open loop system.

Sensitivity due to the variation in \( H(s) \):

from eqn 1.142

\[
\frac{\partial T(s)}{\partial H(s)} = \frac{\frac{G(s)}{1 + G(s)H(s)}^2}{1 + G(s)H(s)}
\]

\[
S_H^T = \frac{H(s)}{T(s)} \frac{\partial T(s)}{\partial H(s)} = \frac{H(s)}{T(s)} \frac{G(s)}{1 + G(s)H(s)} \quad \text{(1.144)}
\]

\[
S_H^T = \frac{-G(s)H(s)}{1 + G(s)H(s)}
\]
From equation 1.143 & 1.142 it is clear that the closed loop system is more sensitive to variations in feedback path parameters than variations in forward path variations.

1.304 Effect of Feedback on Overall Gain

\[ \frac{C(s)}{R(s)} = G(s) \]

The overall transfer function of open loop system shown in fig. 1.118 is

\[ \frac{C(s)}{R(s)} = G(s) \]

The overall transfer function of closed loop system shown in fig. 1.117 is

\[ \frac{C(s)}{R(s)} = \frac{G(s)}{1 + G(s)H(s)} \]

For negative feedback the gain \( G(s) \) is reduced by a factor \( \frac{1}{1 + G(s)H(s)} \). So due to negative feedback overall gain of the system reduces.

1.305. Effect of Feedback on Stability

Consider the open loop system with overall transfer function

\[ G(s) = \frac{K}{s + T} \]

The pole is located at \( s = -T \)

Now, consider closed loop system with unity negative feedback, then overall transfer function is given by

\[ \frac{C(s)}{R(s)} = \frac{K}{s + (T + K)} \]

Now, closed loop pole is located at \( s = -(T + K) \).

Thus, feedback controls the time response by adjusting the location of the poles. The stability depends upon the location of poles. Thus we can say the feedback affects the stability. Feedback can improve the stability or may be harmful to stability if it is not properly design and apply.

1.31. THERMAL SYSTEMS

In thermal systems, there is transfer of heat from one substance to another substance. The thermal system can be analysed in terms of resistance and capacitance. Let us consider a simple thermal system shown in fig. 1.119.

1.31.1. Heat Transfer System

Suppose, there is no heat store in the insulation and water having uniform temperature.

Let, \( \theta_i \) = temperature of inlet water (°C)
\( \theta_o \) = temperature of outlet water (°C)
\( \theta \) = temperature of the surroundings
\( q \) = rate of heat flow from heating elements (J/S)

\( q_i \) = rate of heat flow to the water
\( q_o \) = rate of heat flow through tank insulation
\( C \) = thermal capacitance (J/°C)
\( R \) = thermal resistance (°C/J-S-1)

Rate of heat flow for the water in tank

\[ q_i = \frac{C}{R} \frac{d\theta}{dt} \]

The rate of heat flow from water to the surrounding through insulation

\[ q_o = \frac{\theta_i - \theta}{R} \]

\[ q = q_i + q_o \]

\[ q = \frac{C}{R} \frac{d\theta}{dt} + \frac{\theta_i - \theta}{R} \]

neglect the term \( \frac{\theta}{R} \)

\[ q = \frac{C}{R} \frac{d\theta}{dt} + \theta_i \]

Take laplace transform of (1.149)

\[ Q(s) = sC \theta_o(s) + \theta_i(s) \]

Transfer function

\[ \frac{\theta_o(s)}{Q(s)} = \frac{R}{1 + sRC} \]

\[ \theta_o(s) \quad R \quad 1 + sRC \]

: Time constant of thermal system is \( RC \).

1.31.2. Thermometer

Consider the fig. 1.120 A thermometer is immersed in a tub containing water.

Let \( \theta_i \) = temperature of the water in tub.
\( \theta_o \) = temperature indicated by thermometer then, rate of heat flow to the thermometer

\[ \frac{d\theta}{dt} = \frac{\theta_i - \theta_o}{R} \]

The indicated temperature rises at the rate

\[ \frac{d\theta_o}{dt} = \frac{1}{C} \frac{d\theta}{dt} \]

\[ \frac{d\theta_o}{dt} = \frac{1}{C} \left( \frac{\theta_i - \theta_o}{R} \right) \]

Take the laplace transform

\[ s \theta_o(s) = \frac{1}{RC} \left[ \theta_i(s) - \theta_o(s) \right] \]
1.32. PNEUMATIC SYSTEM

Consider the fig. 1.121. In fig. 1.121 a source is supplying air and the air is stored in vessel.

Let,
- \( P_s \) = Pressure of air of the source \( \text{N/m}^2 \)
- \( P_v \) = Pressure of air in the vessel \( \text{N/m}^2 \)
- \( \Delta P_o \) = Change in air pressure of source
- \( \Delta P_v \) = Change in air pressure of vessel
- \( R \) = Resistance to air flow into the vessel
- \( C \) = Capacitance of the vessel
- \( q \) = Rate of flow of air due to the differential pressure

\[
q = \frac{\Delta P_o - \Delta P_v}{R}
\]

The volume of air stored in vessel increases the pressure inside the vessel.

Volume \( V = C \Delta P_o \)  

\( \theta_o(s) = \frac{1}{1 + sRC} \) \( \theta_o(s) = \text{unit step input} \)

\[
\theta_o(t) = 1 - e^{-t/RC}
\]

Take inverse laplace

\[
\theta_o(t) = \frac{1}{1 + sRC}
\]

\[
\text{Time constant} = RC
\]

Example 1.49. A thermometer has a time constant of 15.33 sec. It is quickly taken from a temperature \( 0^\circ C \) to a water bath having a temp. 100\(^\circ\)C. What temperature will be indicated after 60 sec?

Solution: The thermometer is subjected to a step input of 100\(^\circ\)C

\[
\theta_i(s) = \frac{100}{s}
\]

\[
\theta_i(s) = \frac{100}{s(1 + sRC)}
\]

where \( T \) is the time constant

\[
\theta_i(t) = 100(1 - e^{-t/T})
\]

\[
\theta_i(t) = 98^\circ C
\]

Temperature indicated by thermometer after 60 sec. = 98\(^\circ\)C.
Laplace transform

\[ T(s) = K_T I(s) \]
\[ T(s) = K_0 \theta(s) + s^2 \theta(s) \]
\[ E(s) = I(s) R + K_0 \theta(s) \]
\[ K_T = \frac{K_0 \theta(s) + s^2 \theta(s)}{s^2 + K_T} \]
\[ I(s) = \theta(s) \left( \frac{K_T}{s^2 + K_T} \right) \]
\[ E(s) = R \theta(s) \left( \frac{K_T}{s^2 + K_T} \right) + K_0 \theta(s) \]
\[ \frac{\theta(s)}{E(s)} = \frac{K_T}{s^2 + K_T} \]
\[ \text{Ans.} \]

**SUMMARY**

1. The open loop control system is without feedback. In open loop system the control action is independent of the desired output.
   Closed loop control systems are known as feedback control systems. In closed loop control systems the control action is dependent on the desired output.
2. The elements of the closed loop system are command, reference input, error detection, control element, controlled system and feedback element.
3. Transfer function is defined as the ratio of Laplace transforms of the output to the Laplace transform of input with all initial conditions are zero.
4. Poles are the values of 's' which when substituted in the denominator of a transfer function make the transfer function value as zero.
5. Zeros are the values of 's' which when substituted in the numerator of a transfer function make the transfer function value as zero.
6. The characteristic equation can be obtained by equating the denominator polynomial of the transfer function to zero.
7. The highest power of 's' in the characteristic equation is called the order of the system.
8. The number of poles at the origin defines the type of system.
9. Block diagram is a pictorial representation of the complicated control system.
10. Block diagram can be reduced by reduction rules.
11. The reduction of blocks is possible by the following sequence:
   (i) Reduce the series blocks.
   (ii) Reduce the parallel blocks.
   (iii) Reduce the minor feedback loop.
   (iv) Shift the summing point to the left and take off point to the right, as far as possible.
**Note:** There should be no take off point or summing point between the blocks during the reduction of series blocks.
12. Signal flow graph is applicable to the linear systems. In signal flow graph the variables of the system are represented by separate nodes. The node which has only outgoing branches is known as input node. During construction of signal flow graph represent all variables, summing points and take off points by separate nodes. If a summing point is placed before a take off point represent summing and take off point by a single node. If summing point is placed after take off point, represent summing point and take off point by separate nodes.

13. The resultant transfer function of the system can be obtained from signal flow graph by Mason's gain formula.

\[ T = \frac{\Sigma g_k \Delta_k}{\Delta} \]

where \( T = \) transfer function
\( \Delta = 1 - (\text{sum of all individual loop gain}) + (\text{sum of all possible gain products of two non-touching loops}) - (\text{sum of all possible gain products of three non-touching loops}) + \ldots \)
\( g_k = \text{gain of forward path} \)
\( \Delta_k = \text{the part of } \Delta \text{ not touching the } k^{th} \text{ forward path.} \)

14. The motion takes place along a straight line is known as translational motion.
15. Rotational motion of a body is the motion about a fixed axis.
16. The main elements of any mechanical system with translational motions are mass, spring and friction. The mass \( M \) cannot store energy, hence cannot cause any change in displacement. Spring stores the potential energy which causes in displacement if it is between two moving surfaces. But cannot cause change in displacement, if its one end is connected to a rigid support.
17. The elements of rotational system are inertia \( I \), damping coefficient \( B \) and torsional stiffness \( K \).
18. For every mechanical system, there is analogous electrical system.
19. D'Alembert's principle states that, the algebraic sum of externally applied forces and the forces resisting motion in any given direction is zero.
20. For mechanical network, analogous electrical network can be obtained by using \( f - v \) and \( f - i \) analogy.
21. Force - Voltage analogy: In this method force is analogous to voltage. Similarly

\[ M \rightarrow L, K \rightarrow \frac{1}{C}, B \rightarrow R, \text{displacement } x \text{ charge } q \]

**Force-current analogy:** In this method force is analogous to current

\[ M \rightarrow C, B \rightarrow \frac{1}{R}, K \rightarrow \frac{1}{L}, X \rightarrow \theta \phi \]

**Mechanical Rotational system:**

a. Force-voltage analogy:

\[ T \rightarrow E, I \rightarrow L, B \rightarrow R, K \rightarrow \frac{1}{C}, \theta \rightarrow q \]

b. Force-current analogy:

\[ T \rightarrow I, J \rightarrow C, B \rightarrow \frac{1}{R}, K \rightarrow \frac{1}{L}, \theta \rightarrow \phi \]

**Mechanical Coupling:**

\[ \frac{T_1}{T_2} = \frac{\theta_2}{\theta_1} = \frac{\theta_1}{\theta_2} = \frac{N_2}{N_1} = N_2 \]
EXERCISE

1.1. Determine the transfer function of the given circuit.

1.2. Determine the transfer function.

1.3. Derive the transfer function.

1.4. Determine the transfer function.

1.5. Find out the transfer function if $L = 1\, \Omega$ & $C = 1\, \mu F$.

1.6. Draw the signal flow graph of the electrical circuit shown in fig. 1.79, example 1.28 and find out the transfer function by using the Mason’s gain formula.

1.7. Draw the signal flow graph of the block dia. Shown in fig. 1.91 example 1.32 and determine the transfer function.

1.8. Using Mason’s gain formula, determine the ratio $\frac{C}{R}$.

1.9. Draw the signal flow graph for the following equations

(i) $\frac{d^2y}{dt^2} + 10 \frac{dy}{dt} - 5y = 3x$

(ii) $y'''' + 6y''' + 5y'' + 3y' + 2y = x$

(iii) $\frac{d^2y}{dt^2} + 3y = x$

1.10. Draw the signal flow graph for the following equations

(i) $x_1 + 5x_2 - 2x_1 = 0$

(ii) $x_2 = 2x_2 - 4x_1$

(iii) $x_3 = 2x_2 - 2x_3$

1.11. Find out the transfer function by block reduction method

1.12. For the system shown in fig. obtain $\frac{C(s)}{R(s)}$ by

(i) Block diagram reduction technique

(ii) Signal flow graph

(KNIT, Sultanpur 98-99)
1.13. Determine the overall transfer function relating $C$ and $R$ for the system whose block diag. is given below by block reduction method.

(R.M.I. University Faisalabad LST 2003)

1.14. Determine the transfer function $C/R$ for the system shown in fig. 1.108 (example 1.42) by block reduction technique.

1.15. Determine the transfer function of the given system.

1.16. Draw the electric analog circuit & find $\frac{X_C(s)}{X(s)}$.

1.17. Determine the system equation $\frac{X(s)}{F(s)}$ where $M = 10$ kg, $B = 30$ N/m/s, $K = 20$ N/m.

(R.M.I. University Faisalabad 2003)

1.18. Draw the free body diag. & determine $\frac{X_C(s)}{F(s)}$ for the system shown.

1.19. Draw the block diagram and determine the transfer function.

1.20. Determine the transfer function of the given circuits.

---

**ANSWERS**

1.1. $\frac{s+1}{s+2}$

1.2. $\frac{V_2(s)}{V_1(s)} = \frac{R_2}{(R_1 + R_2) + sL}$

1.3. $\frac{V_2(s)}{V_1(s)} = \frac{SRC}{s^2R^2C^2 + 3SRC + 1}$

1.4. $\frac{V_2(s)}{V_1(s)} = \frac{C_1 + RC_2C_3}{C_1 + C_2 + RC_3}$

1.5. $\frac{V_2(s)}{V_1(s)} = \frac{s^4 + 3s^2 + 1}{s^4 + 3s^2 + 1}$

1.11. $\frac{C(s)}{R(s)} = \frac{G_1G_2}{1 + G_1G_2 + G_2H_2 + G_1G_2H_1}$

1.15. $\frac{C(s)}{R(s)} = \frac{G_1G_2}{1 + G_1G_2 + G_2H_2 + G_1G_2H_1}$

1.17. $\frac{X_C(s)}{F(s)} = \frac{K_1}{10s^2 + 30s + 20}$

1.18. $\frac{X_C(s)}{F(s)} = \frac{K_1}{(s^2M_2 + s^2 + K_1 + K_2)(s^2M_4 + s^2 + K_1 + K_2) - K_1^2}$
**Semi-Objective Type Questions**

- Define open loop and closed loop system.
- Give the advantages and disadvantages of open loop system.
- Define transfer function.
- Define characteristic equation of a transfer function.
- Define poles and zeros of a transfer function.
- Compare translational system with rotational system.
- Define D'Alembert Principle.
- Write the time constant of thermometer.
- What is Mason's gain formula?
- Define signal flow graph.
- Define loop, self loop, path gain, loop gain for SFG.
- How to construct SFG from differential equation.
- What is the effect of feedback on stability.
- Define MIMO system.
- Derive the expression for closed loop transfer function.
- Name the two types of electrical analogies for the mechanical system.
- State the rule for shifting the summing point ahead of a block.
- Define the terms "transmittance" and "non-touching loops" with respect to graph.
- Name the major parts of a closed loop control system.
- What are the basic elements in thermal system?

**Chapter 2**

**Time Domain Analysis**

### 2.1. Introduction

Any system containing energy storing element like inductor, capacitor, mass and inertia etc. possess certain energy. These energy storing elements are the part of the control system and cannot be avoided. If the energy state of the system is disturbed then it takes a certain time to change from one state to another state. This disturbance sometimes occurs at input, sometimes occurs at output and sometimes at both ends. The time required to change from one state to another state is known as transient time and the values of currents and voltages during this period is called transient response. These transient may have oscillations which may be either sustained or decaying in nature. This will depend upon the parameters of the system. For any system we obtain a linear differential equation. The solution of linear differential equation gives the response of the system. Thus, the time response of a control system is divided into two parts (a) transient response (b) steady state response.

![Graph showing input and output response](image)

**Fig. 2.1.**

From the fig. 2.1 it is clear that the transient response is the part of the response which goes to zero as time increases and steady state response is the part of the total response after transient has died. If the steady state response of the output does not match with the input then the system has steady state error.

### 2.2. Test Input Signals for Transient Analysis

For the analysis of time response of a control system, the following input signals are used.
1. Step Function

Consider an independent voltage source is in series with a switch 's'. When switch was open the voltage at terminals 1 - 2 is zero. Mathematically

\[ V(0) = 0 \quad -\infty < t < 0 \]

When switch is closed at \( t = 0 \)

\[ V(t) = K \quad 0 < t < \infty \]

Combining above two equations

\[ V(t) = 0 \quad -\infty < t < 0 \]

\[ = K \quad 0 < t < \infty \]

A unit step function is denoted by \( u(t) \) and is defined as

\[ u(t) = \begin{cases} 0 & ; \quad t \leq 0 \\ 1 & ; \quad 0 < t \end{cases} \]

Laplace Transform: Let \( f(t) \) be defined in the interval \( 0 \leq t \leq \infty \). Laplace transform is obtained by multiplying \( f(t) \) by \( e^{-st} \) and integrate between the limits 0 to \( \infty \).

\[ \mathcal{L}(f(t)) = \int_{0}^{\infty} f(t) e^{-st} \, dt \]

Step function is also called a displacement function. Step function can be described as a sudden application of input signal to a system.

If input is \( R(s) \), then

\[ R(s) = \frac{1}{s} \]

2. Ramp Function

Ramp function starts from origin and increases or decreases linearly with time, as shown in fig. 3.2.

\[ r(t) = \begin{cases} 0 & ; \quad t < 0 \\ Kt & ; \quad t > 0 \end{cases} \]

where 'K' is the slope of the line. For positive value of 'K', the slope is upward and the slope is downwards for negative slope.

Laplace Transform:

\[ \mathcal{L}(r(t)) = \int_{0}^{\infty} r(t) e^{-st} \, dt = \frac{K}{s^2} \]

3. Parabolic Function

The value of \( r(t) \) is zero for \( t < 0 \) and is quadratic function of time for \( t > 0 \). The parabolic function is defined as

\[ r(t) = \begin{cases} 0 & ; \quad t < 0 \\ \frac{Kt^2}{2} & ; \quad t > 0 \end{cases} \]

where 'K' is the constant. For unit parabolic function \( K = 1 \). The unit parabolic function is defined as

\[ r(t) = \begin{cases} 0 & ; \quad t < 0 \\ \frac{t^2}{2} & ; \quad t > 0 \end{cases} \]

Laplace Transform:

\[ \mathcal{L}(r(t)) = \int_{0}^{\infty} r(t) e^{-st} \, dt = \int_{0}^{\infty} \frac{Kt^2}{2} e^{-st} \, dt = \frac{K}{s^3} \]

The parabolic function is shown in fig. 2.4

4. Impulse Function

Consider the fig. 2.5, the first pulse has a width \( T \) and height \( \frac{1}{T} \) such that area of the pulse is

\[ T \times \frac{1}{T} = 1 \]

If we halve the duration and double the amplitude we get second pulse. The area under
the second pulse is also unity. We can say the duration of the pulse approaches zero, the amplitude approaches infinity but the area of the pulse is unity. The pulse for which the duration tends to zero and amplitude tends to infinity is called the impulse function. Impulse function is also known as delta function.

A unit impulse function is defined as

\[ \delta(t) = \begin{cases} 0 & ; \ t \neq 0 \\ \infty & ; \ t = 0 \end{cases} \]

\[ \int_{-\infty}^{\infty} \delta(t) \, dt = 1 \]

Thus, we can say that the impulse function has zero value everywhere except at \( t = 0 \), where its amplitude is infinite. Thus, at \( t = 0 \), unit impulse has infinite amplitude and area equal to unity. Mathematically, an impulse function is the derivative of a step function i.e.

\[ \delta(t) = u(t) \]

\[ \mathcal{L}\{u(t)\} = \frac{1}{s} \]

\[ \mathcal{L}\{\delta(t)\} = s \mathcal{L}\{u(t)\} = \frac{1}{s} \]

\[ \mathcal{L}\{u(t)\} = \frac{1}{s} \]

2.3. TIME RESPONSE OF A FIRST ORDER SYSTEM

Consider a first order system with unity feedback as shown in fig. 2.6.

\[ R(s) \quad \overset{1}{\text{--}} \quad \frac{1}{sT+1} \quad C(s) \]

\[ C(s) = \frac{1}{sT+1} \]

\[ H(s) = 1 \]

\[ C(s) = \frac{1}{sT+1} \]

\[ R(s) = \frac{1}{sT+1} \]

\[ \frac{C(s)}{R(s)} = \frac{1}{sT+1} \]

\[ e(t) = \frac{1}{T} e^{-\frac{t}{T}} \]

When \( t = T \)

\[ C(t) = 1 - e^{-T/T} = 1 - e^{-1} = 0.632 \text{ or } 63.2\% \]

Where, \( T \) is known as time constant and it is defined as the time required for the signal to attain 63.2% of final or steady state value. Time constant indicates how fast the system reaches the final value. Smaller the time constant, faster is the system response. A large time constant corresponds to a sluggish system (slow moving).

Since, the output increases exponentially from zero to final value the slope of the curve at \( t = 0 \) is

\[ \frac{dc}{dt} \bigg|_{t=0} = \frac{1}{T} e^{-t/T} \]

at \( t = 0 \)

\[ \frac{dc}{dt} \bigg|_{t=0} = \frac{1}{T} e^{-0} = \frac{1}{T} \]

From the exponential curve it is clear that, the magnitude of the output response equals to 63.2% of final value in one time constant \( (T) \). In two time constant the magnitude of output reaches 86.4% of final value and approximately 98% in four time constant \( (4T) \). When the actual output reaches within 2% of the desired output it is said that steady state has reached. The time 4 \( T \) is known as settling time (\( T_s \)).

The error is given by

\[ e(t) = r(t) - c(t) = 1 - (1 - e^{-t/T}) = e^{-t/T} \]

Steady state error

\[ \lim_{t \to \infty} e^{-t/T} = 0 \]
2.3.2. Response of the First Order System with Unit Ramp Function

Since, input is the unit ramp \( R(S) = \frac{1}{S} \)

Put the value of \( R(S) \) in equation 2.2

\[
C(S) = \frac{1}{ST+1} \cdot \frac{1}{S^2}
\]

\[
C(S) = \frac{1}{S^2(1+ST)}
\]

After partial fraction of 2.7 can be written as

\[
C(S) = \frac{1-\frac{ST}{S+1}}{S^2} + \frac{T^2}{S^2} + \frac{T}{S+1}
\]

or,

\[
C(S) = \frac{1}{S^2} + \frac{T}{S+1}
\]

Inverse laplace of equation 2.8

\[
L^{-1} \left\{ C(S) \right\} = L^{-1} \left\{ \frac{1}{S^2} + \frac{T}{S+1} \right\}
\]

\[
L^{-1} \left\{ C(S) \right\} = t - T + \frac{1}{e^{\frac{T}{T}}}
\]

The error signal \( e(t) \) will be

\[
e(t) = r(t) - c(t)
\]

\[
e(t) = T - \frac{T}{T} - \frac{T}{e^{\frac{T}{T}}}
\]

\[
e(t) = T \left( 1 - e^{\frac{T}{T}} \right)
\]

Steady state error \( = \lim_{t \to \infty} \left( T - T e^{\frac{T}{T}} \right) = T \)

The steady state error is equal to \( T \), where \( T \) is the time constant of the system. For smaller time constant, steady state error will be small and the speed of the response will increase.

2.3.3. Response of the First Order System with Unit Impulse Function

Input is unit impulse function i.e. \( R(S) = 1 \)

Put the value of \( R(S) \) in equation 2.2

\[
C(S) = \frac{1}{ST+1} \cdot \frac{1}{S}
\]

\[
C(S) = \frac{1}{ST+1}
\]

or,

\[
C(S) = \frac{1}{T} \cdot \frac{1}{S+1}
\]

Inverse laplace of 2.12

\[
L^{-1} \left\{ C(S) \right\} = L^{-1} \left\{ \frac{1}{T} \cdot \frac{1}{S+1} \right\}
\]

\[
L^{-1} \left\{ C(S) \right\} = \frac{1}{T} e^{-\frac{T}{T}}
\]

The curve for \( e(t) \) (2.13) is shown in fig. (2.10).

Compare the equations 2.6, 2.9 and 2.13, that is

For unit ramp input \( R(S) = \frac{1}{S} \)

\[
c(t) = t - T + \frac{T}{e^{\frac{T}{T}}}
\]

For unit step input \( R(S) = \frac{1}{S} \)

\[
c(t) = 1 - e^{\frac{T}{T}}
\]

For unit impulse \( R(S) = 1 \)

\[
c(t) = \frac{1}{T} e^{\frac{T}{T}}
\]

From above three equations, it is clear that the unit step input is the derivative of unit ramp input and unit impulse input is the derivative of unit step input. Thus, the response to the derivative of an input signal can be obtained by differentiating the response of the system to the original signal. This is the property of the linear time-invariant systems.

2.4. TIME RESPONSE OF SECOND ORDER SYSTEM

The block diagram of second order control system is shown in fig. 2.11.

\[
\frac{C(S)}{R(S)} = \frac{G(S)}{1 + G(S)H(S)}
\]

\[
G(S) = \frac{\omega_n^2}{s(s + 2\xi \omega_n)}
\]

\[
H(S) = 1
\]

\[
\frac{C(S)}{R(S)} = \frac{\omega_n^2}{s^2 + 2\xi \omega_n s + \omega_n^2}
\]

2.4.1. Time Response of Second Order System with Unit Step Input

From equation (2.14)

\[
C(S) = \frac{\omega_n^2}{s^2 + 2\xi \omega_n s + \omega_n^2} \cdot R(S)
\]

For unit step input \( R(S) = \frac{1}{s} \)

\[
c(s) = \frac{1}{s} \cdot \frac{\omega_n^2}{s^2 + 2\xi \omega_n s + \omega_n^2}
\]

Replace \( s^2 + 2\xi \omega_n s + \omega_n^2 \) by \( (s + \xi \omega_n)^2 + \omega_n^2(1 - \xi^2) \)

\[
c(S) = \frac{1}{s + \xi \omega_n s + \omega_n^2(1 - \xi^2)}
\]

Break the Equation 2.16 by partial fraction, put \( \omega_n^2 = \omega_n^2 - \omega_n^2 \)

\[
\frac{\omega_n^2}{s^2 + 2\xi \omega_n s + \omega_n^2} = \frac{A}{s} + \frac{B}{s + \omega_n(1 - \xi^2)}
\]
Multiply equation 2.17 by \( s \) and put \( s = 0 \)

\[
\frac{\omega_n^2}{s^2 + 2\zeta \omega_n s + \omega_n^2} = A
\]

Put

\[
\omega_n^2 = \omega_n^2 \left( 1 - \zeta^2 \right) = \omega_n^2 - \omega_n^2 s^2
\]

\[ A = \frac{\omega_n^2}{s^2 \omega_n^2 + 2\zeta \omega_n \omega_n^2 + \omega_n^2} = 1 \quad \therefore \quad A = 1 \]

Multiply equation 2.17 both side by \((S + \zeta \omega_n)^2 + \omega_n^2\) and put

\[
S = -\xi \omega_n + j\omega_n
\]

\[
\frac{\omega_n^2}{s^2 + 2\zeta \omega_n s + \omega_n^2} = B
\]

\[
B = \frac{-\omega_n^2}{-\xi \omega_n - j\omega_n} = \frac{\omega_n^2}{\xi \omega_n + j\omega_n}
\]

\[
= \frac{\omega_n^2 \left( 2\xi \omega_n - j\omega_n \right)}{(\xi \omega_n + j\omega_n) (\xi \omega_n - j\omega_n)} = \left( \xi \omega_n - j\omega_n \right)
\]

\[
p = -j\omega_n = s + \zeta \omega_n
\]

\[
B = \left( \xi \omega_n + s + \zeta \omega_n \right) = \left( S + 2\zeta \omega_n \right)
\]

Equation 2.36 can be written as

\[
C(s) = \frac{1}{s} \left( \frac{s + 2\zeta \omega_n}{(s + \zeta \omega_n)^2 + \omega_n^2} \right) = \left[ \frac{1}{s} \left( \frac{s + \zeta \omega_n + \zeta \omega_n}{(s + \zeta \omega_n)^2 + \omega_n^2} \right) \right]
\]

or,

\[
C(s) = \frac{1}{s} \left( \frac{s + \zeta \omega_n + \zeta \omega_n}{(s + \zeta \omega_n)^2 + \omega_n^2} \right) = \frac{1}{s} \left( \frac{s + \zeta \omega_n}{(s + \zeta \omega_n)^2 + \omega_n^2} \right)
\]

Laplace inverse of equation (2.18)

\[ C(t) = 1 - \left[ e^{-\frac{\zeta \omega_n}{\sqrt{1 - \zeta^2}}} \cos \omega_n t + \frac{\zeta \omega_n}{\sqrt{1 - \zeta^2}} \sin \omega_n t \right] \]

Put

\[
\omega_d = \omega_n \sqrt{1 - \zeta^2}
\]

\[
C(t) = 1 - e^{-\frac{\zeta \omega_n}{\sqrt{1 - \zeta^2}}} \left[ \cos \omega_n t + \frac{\zeta \omega_n}{\sqrt{1 - \zeta^2}} \sin \omega_n t \right]
\]

\[
\sqrt{1 - \zeta^2} = \sin \phi
\]

\[
\cos \phi = \xi, \quad \tan \phi = \sqrt{\frac{1 - \zeta^2}{\xi}}
\]

\[
C(t) = 1 - e^{-\phi/\sqrt{1 - \zeta^2}} \left[ \sin \phi \cos \omega_n t + \cos \phi \sin \omega_n t \right] = 1 - \frac{e^{-\phi/\sqrt{1 - \zeta^2}}}{\sin \left( \omega_n t + \phi \right)} \sin \left( \omega_n t + \phi \right)
\]

Put the values of \( \omega_d \) & \( \phi \) i.e.

\[
\omega_d = \omega_n \sqrt{1 - \zeta^2}, \quad \phi = \tan^{-1} \frac{1 - \zeta^2}{\xi}
\]

\[ C(t) = 1 - e^{-\omega_d t/\sqrt{1 - \zeta^2}} \sin \left( \omega_n \sqrt{1 - \zeta^2} t + \tan^{-1} \frac{1 - \zeta^2}{\xi} \right) \]

The error signal for the system

\[ e(t) = r(t) - c(t) \]

\[ e(t) = \frac{e^{-\omega_d t}}{\sqrt{1 - \zeta^2}} \sin \left( \omega_n \sqrt{1 - \zeta^2} t + \tan^{-1} \frac{1 - \zeta^2}{\xi} \right) \]

The steady state value of \( c(t) \)

\[ c_{ss} = \lim_{t \to \infty} c(t) = 1 \]

Therefore at steady state there is no error between input & output.

\( \omega_n \) = natural frequency of oscillation or undamped natural frequency

\( \omega_d \) = damped frequency of oscillation

\( \xi \) = damping factor or actual damping or damping coefficient.

(a) Underdamped case \((0 < \xi < 1)\)

From the expression 2.20, it is clear that the time constant is \( 1/\xi \omega_n \), and the response has damped oscillation with overshoot and undershoot. Such response is known as underdamped response. The response is shown in fig. 2.12.

(b) When \( \xi = 0 \), undamped case

The expression 2.20 will be

\[ C(t) = 1 - \sin \left( \omega_n t + \phi \right) \]

\[ C(t) = 1 - \cos \omega_n t \]

Thus at \( \omega_n \), the system will oscillate (with \( \xi = 0 \)). The damped frequency always less than the undamped frequency \( (\omega_n) \) because of factor \( \xi \). If the system having certain value of \( \xi \) then it is not possible to measure undamped natural frequency experimentally. The observed frequency is damped frequency \((\omega_d)\) which is equal to \( \omega_n \sqrt{1 - \zeta^2} \). The response is shown in fig. 2.13.

(c) \( \xi = 1 \) critically damped case:

Put \( \xi = 1 \) in equation 2.15
\[ C(s) = \frac{1}{s} \frac{s^2 + \omega_n^2}{s^2 + 2 \xi \omega_n s + \omega_n^2} \]

or,
\[ C(s) = \frac{1}{s} \frac{\omega_n^2}{(s + \omega_n)^2} \]

Break the equation 2.23 by partial fraction
\[ \frac{\omega_n^2}{s(s + \omega_n)^2} = \frac{A}{s} + \frac{B}{s + \omega_n} + \frac{C}{(s + \omega_n)^2} \]

Multiply both the sides by \((s + \omega_n)^2\) and put \(s = -\omega_n\)
\[ \frac{\omega_n^2}{s(s + \omega_n)^2} = \frac{A}{s} + \frac{B}{s + \omega_n} + \frac{C}{(s + \omega_n)^2} = \frac{\omega_n^2}{s(s + \omega_n)^2} \]

Differentiate eqn 2.25 & put \(s = -\omega_n\)
\[ \frac{\omega_n^2}{s(s + \omega_n)^2} = \frac{A}{s} + \frac{B}{s + \omega_n} + \frac{C}{(s + \omega_n)^2} \]

\[ A = 1 \]

\[ \frac{\omega_n^2}{s(s + \omega_n)^2} = \frac{1}{s} + \frac{B}{s + \omega_n} + \frac{C}{(s + \omega_n)^2} \]

Inverse laplace of equation 2.25
\[ c(t) = \frac{1}{s} \left( \frac{1}{s} \right) = \frac{1}{s} \]

\[ c(t) = \frac{1}{s} \left( \frac{1}{s} \right) = \frac{1}{s} \]

\[ c(t) = \frac{1}{s} \left( \frac{1}{s} \right) = \frac{1}{s} \]

Put all values in eqn. (2.27)
\[ c(t) = 1 - e^{-\alpha t} - t e^{-\alpha t} \]
\[ = 1 - e^{-\alpha t} (1 + t) \]

From the expression 2.20, it is clear that \(\xi \omega_n\) is the actual damping for \(\xi = 1\), the actual damping is \(\omega_n\). This actual damping when \(\xi = 1\) is known as CRITICAL DAMPING. At the value of critical damping the oscillations just disappeared. The ratio of actual damping to the critical damping is known as damping ratio \(\zeta\).

Actual damping \(\zeta\)
Critical damping \(\zeta\)

From eqn 2.28, the time constant of the system is \(1/\xi \omega_n\).

The response of the equation 2.28 is shown in fig 2.14.

\[(d) \ \xi > 1 \text{ overdamped case}\]

For \(\xi > 1\), eqn 2.16 can be written as

\[ c(t) = \frac{1}{s(\xi \omega_n - \omega_n)} \left( \frac{\omega_n^2}{s^2 + \xi \omega_n s + \omega_n^2} \right) - \frac{\omega_n^2}{s^2 + \xi \omega_n s + \omega_n^2} (\xi^2 - 1) \]

\[ c(t) = \frac{1}{s(\xi \omega_n - \omega_n)} \left( \frac{\omega_n^2}{s^2 + \xi \omega_n s + \omega_n^2} \right) - \frac{\omega_n^2}{s^2 + \xi \omega_n s + \omega_n^2} (\xi^2 - 1) \]

\[ c(t) = \frac{1}{s(\xi \omega_n - \omega_n)} \left( \frac{\omega_n^2}{s^2 + \xi \omega_n s + \omega_n^2} \right) - \frac{\omega_n^2}{s^2 + \xi \omega_n s + \omega_n^2} (\xi^2 - 1) \]

Break the equation (2.30) by partial fraction

Equation 2.30 can be written as

\[ c(t) = \frac{1}{s(\xi \omega_n - \omega_n)} \left( \frac{\omega_n^2}{s^2 + \xi \omega_n s + \omega_n^2} \right) - \frac{\omega_n^2}{s^2 + \xi \omega_n s + \omega_n^2} (\xi^2 - 1) \]

Multiply both the sides by \(s\) and put \(s = -\omega_n\)

\[ A = 1 \]

Multiply both the sides of eqn 2.31 by \(s + \xi \omega_n + \omega_n\) and put \(s = -\xi \omega_n - \omega_n\)

\[ c(t) = \frac{1}{s(\xi \omega_n - \omega_n)} \left( \frac{\omega_n^2}{s^2 + \xi \omega_n s + \omega_n^2} \right) - \frac{\omega_n^2}{s^2 + \xi \omega_n s + \omega_n^2} (\xi^2 - 1) \]

Put \(\omega_n^2 = \omega_n^2 (\xi^2 - 1)\) and simplify for \(B\)

\[ \omega_n^2 = \omega_n^2 (\xi^2 - 1) \]

\[ B = \frac{1}{2 \xi^2 - 1 - 2(\xi^2 - 1)} = \frac{1}{2 \xi^2 - 1 - 2(\xi^2 - 1)} \]

\[ B = \frac{1}{2 \xi^2 - 1 - 2(\xi^2 - 1)} \]

\[ B = \frac{1}{2 \xi^2 - 1 - 2(\xi^2 - 1)} \]

\[ B = \frac{1}{2 \xi^2 - 1 - 2(\xi^2 - 1)} \]
Similarly,
\[
C = \frac{1}{2N_1^2 - 1} \left[ N_1 + \sqrt{N_1^2 - 1} \right]
\]

Put the values of \( N_1 \), \( N_2 \) and \( C \) in (2.31)
\[
c(t) = \frac{1}{2N_1^2 - 1} \left[ N_1 + \sqrt{N_1^2 - 1} \right] \left[ s + N_1 \omega_n + \omega_n \right] \frac{1}{2N_1^2 - 1} \left[ N_1 - \sqrt{N_1^2 - 1} \right] \left[ s + N_1 \omega_n - \omega_n \right]
\]

Put the value of \( \omega_n \)
\[
c(t) = \frac{1}{2N_1^2 - 1} \left[ N_1 + \sqrt{N_1^2 - 1} \right] \left[ s + \omega_n \right] \frac{1}{2N_1^2 - 1} \left[ N_1 - \sqrt{N_1^2 - 1} \right] \left[ s + \omega_n \right]
\]

Inverse Laplace of (2.32)
\[
c(t) = \frac{1}{2N_1^2 - 1} e^{t \sqrt{N_1^2 - 1} \omega_n} - \frac{1}{2N_1^2 - 1} e^{-t \sqrt{N_1^2 - 1} \omega_n}
\]

From equation (2.33) we get two time constant
\[
T_1 = \frac{1}{\xi \left( \xi + \sqrt{\xi^2 - 1} \right) \omega_n}, \quad T_2 = \frac{1}{\xi \left( \xi - \sqrt{\xi^2 - 1} \right) \omega_n}
\]

From eq. (2.33) we observe that when \( \xi \) is greater than one there are two exponential terms, the first term has a time constant \( T_1 \), which is smaller than the time constant of other exponential terms (having time constant \( T_2 \)). In other words, we can say that the first exponential term decaying faster than the other exponential terms. So, for time response we can neglect it, then
\[
c(t) = \frac{1}{2N_1^2 - 1} e^{t \sqrt{N_1^2 - 1} \omega_n}
\]

and, time constant
\[
T = \frac{1}{\xi \left( \xi - \sqrt{\xi^2 - 1} \right) \omega_n}
\]

For different values of \( \xi \) the curves of \( c(t) \) is shown in fig. 2.15. From the curve it is clear that the overdamped systems are sluggish.

2.4.2. Location of Roots of Characteristic Equation and Time Response

The characteristic equation of
\[
\frac{C(s)}{R(s)} = \frac{\omega_n^2}{s^2 + 2N_1 \omega_n s + \omega_n^2}
\]
is
\[
s^2 + 2N_1 \omega_n s + \omega_n^2 = 0
\]
The roots of eq. 2.36
\[
S_1 = -N_1 \omega_n + jN_1 \omega_n \sqrt{1 - \xi^2}
\]
\[
S_2 = -N_1 \omega_n - jN_1 \omega_n \sqrt{1 - \xi^2}
\]
The real part of the roots \(-N_1 \omega_n\) represents the damping and imaginary part \(N_1 \omega_n \sqrt{1 - \xi^2}\) represents the damped frequency.

From the fig. 2.16
1. \( \omega_n \) is the distance of root from origin.
2. \( \cos \theta = \frac{\xi \omega_n}{\omega_n} = \xi, \ i.e. \theta = \cos^{-1} \xi \) (when the roots are in left-half of s-plane).
3. \( \omega_d \) is the imaginary part of the root and is known as damped frequency or conditional frequency.
2.5. TRANSIENT RESPONSE SPECIFICATIONS OF SECOND ORDER SYSTEM

The performance of a control system are expressed in terms of the transient response to a unit step input. The transient response of a control system to a unit step input is easy to generate. The transient response of a control system is a fundamental concept in control systems. Consider a second-order system with unit step input and initial conditions. The initial conditions of the system are assumed to be zero.

The following are the commonly occurring transient response characteristics.

1. **Delay Time** ($t_d$): The delay time is the time required for the response to reach 50% of the final value in first time.

2. **Rise Time** ($t_r$): It is the time required for the response to rise from 10% to 90% of its final value.

3. **Peak Time** ($t_p$): The peak time is the time required for the response to reach the first peak of the time response or first peak overshoot.

4. **Maximum Overshoot** ($M_o$): It is the normalized difference between the peak of the transient response and steady output. The maximum percent overshoot is defined by

   \[ \text{Maximum percent overshoot} = \frac{C(t_p) - C(\infty)}{C(\infty)} \times 100 \]

5. **Settling Time** ($t_s$): The settling time is the time required for the response to reach and stay within the specified range (2% to 5%) of its final value.

6. **Steady State Error** ($e_s$): It is the difference between actual output and desired output as time $t$ tends to infinity.

\[ e_s = \lim_{t \to \infty} [r(t) - C(t)] \]

**Expression for Rise Time** ($t_r$):

From the expression (2.20)

\[ C(t) = 1 - \frac{e^{-\omega_n t}}{\sqrt{1 - \xi^2}} \sin \left[ \left( \frac{\omega_n \sqrt{1 - \xi^2}}{\sqrt{1 - \xi^2}} \right) t + \phi \right] \]

where $\phi = \tan^{-1} \left( \frac{\xi}{\sqrt{1 - \xi^2}} \right)$

Let response reaches 100% of desired value. Put $r(t) = 1$

\[ 1 = 1 - \frac{e^{-\omega_n t}}{\sqrt{1 - \xi^2}} \sin \left[ \left( \frac{\omega_n \sqrt{1 - \xi^2}}{\sqrt{1 - \xi^2}} \right) t + \phi \right] \]

or,

\[ \frac{1}{\sqrt{1 - \xi^2}} \sin \left( \frac{\omega_n \sqrt{1 - \xi^2}}{\sqrt{1 - \xi^2}} \right) t + \phi = \pi \]

Since, $e^{-\omega_n t} = 0$

\[ \sin \left[ \left( \frac{\omega_n \sqrt{1 - \xi^2}}{\sqrt{1 - \xi^2}} \right) t + \phi \right] = 0, \text{ or } \sin \left[ \left( \frac{\omega_n \sqrt{1 - \xi^2}}{\sqrt{1 - \xi^2}} \right) t + \phi \right] = \sin n\pi \]

Put $n = 1$

\[ \left( \frac{\omega_n \sqrt{1 - \xi^2}}{\sqrt{1 - \xi^2}} \right) t + \phi = \pi \]

or,

\[ t_r = \frac{\pi - \phi}{\omega_n \sqrt{1 - \xi^2}} \]

Expression for Peak Time ($t_p$):

since, $C(t) = 1 - \frac{e^{-\omega_n t}}{\sqrt{1 - \xi^2}} \sin \left[ \left( \frac{\omega_n \sqrt{1 - \xi^2}}{\sqrt{1 - \xi^2}} \right) t + \phi \right]$

For maximum

\[ \frac{dc(t)}{dt} = 0 \]

\[ \frac{dc(t)}{dt} = \frac{e^{-\omega_n t}}{\sqrt{1 - \xi^2}} \cos \left[ \left( \frac{\omega_n \sqrt{1 - \xi^2}}{\sqrt{1 - \xi^2}} \right) t + \phi \right] \omega_n \sqrt{1 - \xi^2} + \sin \left[ \left( \frac{\omega_n \sqrt{1 - \xi^2}}{\sqrt{1 - \xi^2}} \right) t + \phi \right] \]

\[ \frac{\xi \omega_n}{\sqrt{1 - \xi^2}} e^{-\omega_n t} \]

\[ \text{(2.38)} \]
Since $e^{-\omega_n t} = 0$

$E_p$ 2.38 can be written as

$$\cos\left(\omega_n \sqrt{1-\xi^2} t + \phi\right) \sqrt{1-\xi^2} = \sin\left(\omega_n \sqrt{1-\xi^2} t + \phi\right) \xi.$$  

Put

$$\sqrt{1-\xi^2} = \sin \phi \quad \text{&} \quad \xi = \cos \phi.$$

$E_p$ 2.39 becomes

$$\cos\left(\omega_n \sqrt{1-\xi^2} t + \phi\right) \sin \phi = \sin\left(\omega_n \sqrt{1-\xi^2} t + \phi\right) \cos \phi$$

or

$$\sin\left(\omega_n \sqrt{1-\xi^2} t + \phi\right) \cos \phi - \cos\left(\omega_n \sqrt{1-\xi^2} t + \phi\right) \sin \phi = 0$$

or,

$$\sin\left(\omega_n \sqrt{1-\xi^2} t + \phi\right) \cos \phi = 0$$

the time to various peaks

$$\left(\omega_n \sqrt{1-\xi^2} t \right) = n \pi$$

where $n = 0, 1, 2, 3, \ldots$

Maximum overshoot identified by putting $n = 1$, therefore the peak time to the first overshoot

$$t_p = \frac{\pi}{\omega_n \sqrt{1-\xi^2}}$$

The first minimum (undershoot) occurs at $n = 2$

$$t_{min} = \frac{2\pi}{\omega_n \sqrt{1-\xi^2}}$$

Expression for Maximum Overshoot $M_p$

$$C(t) = 1 - e^{-\omega_n t} \sin\left(\omega_n \sqrt{1-\xi^2} t + \phi\right)$$

Maximum overshoot occurs at peak time i.e. $t = t_p$

Put

$$t = t_p = \frac{\pi}{\omega_n \sqrt{1-\xi^2}}$$

in eqn 2.42

$$C(t) = 1 - e^{-\omega_n t} \sin\left(\omega_n \sqrt{1-\xi^2} \frac{\pi}{\omega_n \sqrt{1-\xi^2}} + \phi\right)$$

$$= 1 - e^{-\omega_n t} \sin (\pi + \phi)$$

Since

$$\phi = \tan^{-1} \frac{\sqrt{1-\xi^2}}{\xi}$$

and

$$\sin (\pi + \phi) = -\sin \phi$$

$$C(t) = 1 + e^{-\omega_n t} \frac{\sqrt{1-\xi^2}}{-\xi} \sin \phi$$

$$C(t)_{max} = 1 + e^{-\omega_n t} \frac{\sqrt{1-\xi^2}}{-\xi}$$

$$M_p = C(t)_{max} - 1$$

$$M_p = e^{-\omega_n t} \frac{\sqrt{1-\xi^2}}{-\xi}$$

and

$$\% M_p = e^{-\omega_n t} \frac{\sqrt{1-\xi^2}}{-\xi} \times 100$$

Setting Time $t_s$:

As shown in the fig 2.17, the curves for $1 \pm e^{-\omega_n t}$ are the envelope curves of the transient response for unit step input. The time constant of these envelope curves is $\frac{1}{\xi \omega_n}$. The speed of the decay depends upon the time constant. The settling time for a second order system is approximately four times the time constant ($1/\xi \omega_n$)

$$t_s = \frac{4}{\xi \omega_n}$$

For under damped system, the settling time $t_s$ becomes large because of sluggish start. From 2.45, the settling time is inversely proportional to the product of $\xi$ and $\omega_n$. So for permissible maximum overshoot, the value of $\xi$ is known therefore the settling time can be determined by undamped natural frequency $\omega_n$

Example 2.1. When a second order control system is subjected to a unit step input, the values of $\xi = 0.5$ and $\omega_n = 6 \text{ rad/sec}$. Determine the rise time, peak time, settling time and peak overshoot.

Solution : Given that

$$\xi = 0.5 \quad \omega_n = 6 \text{ rad/sec}.$$

1. RISE TIME:

$$t_r = \frac{\pi - \tan^{-1} \frac{\sqrt{1-\xi^2}}{\xi}}{\omega_n \sqrt{1-\xi^2}}$$

$$t_r = \frac{\pi - \tan^{-1} \frac{\sqrt{1-(0.5)^2}}{0.5}}{6 \sqrt{1-(0.5)^2}}$$

$$t_r = \frac{3.14 - 1.047}{5.19} = 0.403 \text{ sec.}$$
2. PEAK TIME:

\[ t_p = \frac{\pi}{\omega_n \sqrt{1 - \xi^2}} \]

\[ t_p = \frac{\pi}{6\sqrt{1 - (0.5)^2}} = 0.605 \text{ sec.} \]

3. SETTLING TIME:

\[ t_s = \frac{4}{\xi \omega_n} \]

\[ t_s = \frac{4}{0.5 \times 6} = 1.33 \text{ sec.} \]

4. MAXIMUM OVERTURE:

\[ M_p = \frac{1}{\sqrt{1 - \xi^2}} \times 100 = \frac{1}{\sqrt{1 - (0.5)^2}} \times 100 \]

\[ M_p = 0.163 \times 100 = 16.3 \% \]

2.6. TIME RESPONSE OF SECOND ORDER SYSTEM WITH UNIT IMPULSE INPUT

Since, input is unit impulse \( R(S) = 1 \)

\[ C(t) = \frac{\omega_n^2}{s^2 + 2\xi\omega_n s + \omega_n^2} \]

Break the equation 2.46 by partial fraction

\[ \frac{\omega_n^2}{s^2 + 2\xi\omega_n s + \omega_n^2} = \frac{K_1}{s - s_1} + \frac{K_2}{s - s_2} \]

characteristic \( s^2 + 2\xi\omega_n s + \omega_n^2 = 0 \)

Roots of the characteristic equation: \( s_1 = -\xi\omega_n + j\omega_n \sqrt{1 - \xi^2} \)

\( s_2 = -\xi\omega_n - j\omega_n \sqrt{1 - \xi^2} \)

Put

\[ s_1 = \omega_n \sqrt{1 - \xi^2} \]

\( s_2 = -\xi\omega_n \)

From equation (2.47)

\[ K_1 = \frac{\omega_n^2}{s - s_1} = \frac{\omega_n^2}{-\xi\omega_n + j\omega_n \sqrt{1 - \xi^2}} = \frac{\omega_n^2}{2j\omega_n} \]

\[ K_2 = \frac{\omega_n^2}{s - s_2} = \frac{\omega_n^2}{-\xi\omega_n - j\omega_n \sqrt{1 - \xi^2}} = \frac{\omega_n^2}{2j\omega_n} \]

Therefore equation (2.47) becomes

\[ C(s) = \frac{\omega_n^2}{s^2 + 2\xi\omega_n s + \omega_n^2} = \frac{\omega_n^2}{2j\omega_n} \left[ \frac{1}{s - s_1} - \frac{1}{s - s_2} \right] \]

Inverse laplace of eq. 2.48

\[ c(t) = \frac{\omega_n^2}{2j\omega_n} \left[ e^{s_1 t} - e^{s_2 t} \right] \]

\[ = \frac{\omega_n^2}{2j\omega_n} \left[ e^{\omega_n \sqrt{1 - \xi^2} t} - e^{-\xi\omega_n t} \right] \]

\[ = \frac{\omega_n^2}{2j\omega_n} \left[ \frac{1}{s - s_1} - \frac{1}{s - s_2} \right] \]

\[ = \frac{\omega_n^2}{2j\omega_n} \sin \left( \omega_n \sqrt{1 - \xi^2} t \right) \]

Put the value of \( \omega_d \) in eqn. 2.49

\[ C(t) = \frac{\omega_n^2}{\sqrt{1 - \xi^2}} \sin \left( \omega_n \sqrt{1 - \xi^2} t \right) \]

\[ C(t) = \frac{\omega_n^2}{\sqrt{1 - \xi^2}} \sin \left( \omega_n \sqrt{1 - \xi^2} t \right) \]

(a) \( \xi < 1 \):

\[ C(t) = \frac{\omega_n^2}{\sqrt{1 - \xi^2}} \sin \left( \omega_n \sqrt{1 - \xi^2} t \right) \]

\[ C(t) = \frac{\omega_n^2}{\sqrt{1 - \xi^2}} \sin \left( \omega_n \sqrt{1 - \xi^2} t \right) \]

(b) \( \xi = 1 \):

\[ C(t) = \frac{\omega_n^2}{\sqrt{1 - \xi^2}} \sin \left( \omega_n \sqrt{1 - \xi^2} t \right) \]

\[ C(t) = \frac{\omega_n^2}{\sqrt{1 - \xi^2}} \sin \left( \omega_n \sqrt{1 - \xi^2} t \right) \]

(c) \( \xi > 1 \):

Eq. 2.46 can be written

\[ C(S) = \frac{\omega_n^2}{(s + \xi\omega_n)^2 - \omega_n^2 (\xi^2 - 1)} \]

Break the eqn. 2.53 by partial fraction put \( \omega_d^2 = \omega_n^2 (\xi^2 - 1) \)

\[ \frac{\omega_n^2}{(s + \xi\omega_n)^2 - \omega_n^2} = \frac{A}{s + \xi\omega_n - \omega_n} + \frac{B}{s + \xi\omega_n + \omega_n} \]

\[ A = \frac{\omega_n^2}{(s + \xi\omega_n + \omega_n)(s + \xi\omega_n - \omega_n)} \]

\[ B = \frac{\omega_n^2}{2(\xi^2 - 1)} \]

From (2.54)

\[ A = \frac{\omega_n^2}{s + \xi\omega_n - \omega_n} \]

\[ B = \frac{\omega_n^2}{2(\xi^2 - 1)} \]

Similarly,

\[ A = \frac{\omega_n^2}{s + \xi\omega_n + \omega_n} \]

\[ B = \frac{\omega_n^2}{2(\xi^2 - 1)} \]
\[ C(S) = \frac{\omega_n^4}{\left(s + (\omega_n \sqrt{1 - \xi^2})\right)^4} \]
\[ = \frac{-\omega_n}{2\sqrt{\xi^2 - 1}} \frac{1}{s + \xi \omega_n + \omega_n^2} + \frac{\omega_n}{2\sqrt{\xi^2 - 1}} \frac{1}{s + \xi \omega_n - \omega_n^2} \]

Take inverse Laplace
\[ C(t) = \frac{\omega_n}{2\sqrt{\xi^2 - 1}} \left[ e^{-(\xi \omega_n - \omega_n) t} + e^{-(\xi \omega_n + \omega_n) t} \right] \]
\[ \therefore C(t) = \frac{\omega_n}{2\sqrt{\xi^2 - 1}} \left[ e^{-(\xi \omega_n - \omega_n) t} - e^{-(\xi \omega_n + \omega_n) t} \right] \]
\[ \therefore C(t) = \frac{\omega_n}{2\sqrt{\xi^2 - 1}} \left[ e^{-(\xi \omega_n - \omega_n) t} - e^{-(\xi \omega_n + \omega_n) t} \right] \]
... (2.56)

Let the maximum overshoot for the unit impulse response of the underdamped system occur at
\[ t = t_f \]
\[ \frac{dC(t)}{dt} = 0 \]
Differentiate eq. 2.51 and put \( a_d = \omega_n \sqrt{1 - \xi^2} \)
\[ \frac{\omega_n}{\sqrt{1 - \xi^2}} \left[ e^{-\omega_n t} \cos \omega_d t \omega_n + e^{-\omega_n t} \sin \omega_d t \right] = 0 \]
\[ e^{-\omega_n t} \neq 0 \]
\[ \omega_n \cos \omega_d t = \frac{\omega_n}{\sqrt{1 - \xi^2}} \sin \omega_d t \]
\[ \tan \omega_d t = \frac{\omega_n}{\omega_d} \frac{1}{\sqrt{1 - \xi^2}} \]
\[ \omega_d t = \tan^{-1} \frac{1}{\sqrt{1 - \xi^2}} \frac{\omega_n}{\omega_d} \]
\[ t_f = \frac{\omega_n}{\sqrt{1 - \xi^2}} \]

Expression for maximum overshoot.
Since \( \tan \omega_d t = \frac{1}{\sqrt{1 - \xi^2}} \), then \( \sin \omega_d t = \omega_n \sqrt{1 - \xi^2} \)
\[ C(t) = \frac{\omega_n}{\sqrt{1 - \xi^2}} e^{\omega_n \sqrt{1 - \xi^2} t} \]

2.7. TIME RESPONSE OF SECOND ORDER SYSTEM WITH UNIT RAMP INPUT
\[ \frac{C(S)}{R(S)} = \frac{\omega_n^2}{s^2 + 2\xi \omega_n s + \omega_n^2} \]

Input
\[ R(S) = \frac{1}{s} \]

\[ C(S) = \frac{\omega_n^2}{s^2 (s^2 + 2\xi \omega_n s + \omega_n^2)} \]

Break the eqn. 2.60 by partial fraction
\[ \frac{\omega_n^2}{s^2 (s^2 + 2\xi \omega_n s + \omega_n^2)} = \frac{A}{s} + \frac{B}{s^2} + \frac{cs + D}{s^2 + 2\xi \omega_n s + \omega_n^2} \]
Equating the coefficient of \( s^2 \):
\[ \frac{A}{s} + \frac{B}{s^2} = \omega_n^2 \]
Equating the coefficient of \( s^3 \):
\[ A = 0 \]
Equating the coefficient of \( s^4 \):
\[ B = 1, D = -4\xi^2 \]
From equations (2.62), (2.63), (2.64), & (2.65)
\[ A = \frac{2s}{\omega_n}, C = \frac{2s}{\omega_n} \]
\[ B = 1, D = -4\xi^2 \]
Put all these values in eqn. (2.61)
\[ C(S) = \frac{-\omega_n}{s^2 + \frac{2s}{\omega_n} s + \frac{2s^2}{\omega_n} - 1} \]
\[ = \frac{1}{s^2} - \frac{2s}{\omega_n} + \frac{2s}{\omega_n} (s + \xi \omega_n)^2 + \frac{2s^2}{\omega_n} (s + \xi \omega_n)^2 + \frac{4s^2}{\omega_n} (s + \xi \omega_n)^2 + \frac{4s^2}{\omega_n} (s + \xi \omega_n)^2 + \frac{4s^2}{\omega_n} (s + \xi \omega_n)^2 \]
2.8. The Response of Higher Order Control System

Example 2.1. A second order system has a transfer function given by

\[ C(s) = \frac{10}{s^2 + 2s + 25} \]

If the system initially at rest is subjected to a unit step input at \( t = 0 \), the second peak in the response will occur at \( t = \) sec, \( t = \) sec, \( t = \) sec, \( t = \) sec, \( t = \) sec.

The Laplace transform of the system when a unit step is applied at \( t = 0 \) is

\[ C(s) = 10s^2 + 20s + 50 \]

This is the required result.

Example 2.2. The closed loop transfer function is given by

\[ \frac{C(s)}{R(s)} = \frac{K(s + 1)}{s^2 + 3s + 2} \]

The input function and unit step input, the response of the system is

\[ C(s) = \frac{10s^2 + 20s + 50}{s^2 + 3s + 2} \]

This is the required result.

Example 2.3. The closed loop transfer function is given by

\[ T(s) = \frac{K(s + 1)}{s^2 + 3s + 2} \]

For time response take inverse Laplace of each part of above "s".
Example 2.4. Consider the system as shown in Fig. 2.18. Determine the value of $s$ such that the damping ratio is 0.5. Also, obtain the values of rise time and maximum overshoot $M_p$ in its step response.

Solution:

\[ C(S) = \frac{R(S)}{1 + \frac{G}{s(s + 3)}} \]

\[ C(S) = \frac{16}{s(s + 0.8)} \]

\[ \frac{C(S)}{R(S)} = \frac{16}{s(s + 0.8)} \]

\[ \frac{16}{s(s + 0.8)} \]

\[ s^2 + 0.8s + 16 \]

\[ s^2 + 0.8s + A = 0 \]

\[ 2\zeta \omega_n = 0 \]

\[ \omega_n = \frac{\sqrt{G}}{\zeta} \]

\[ \zeta = 0.6, \text{ (assume)} \]

\[ 2 \times 0.6 \sqrt{G} = 3 \]

\[ G = 6.25 \]

\[ G' = 2G = 2 \times 6.25 = 12.5 \]

\[ \omega_n = \frac{\sqrt{12.5}}{3.53} \text{ rad/sec.} \]

\[ \zeta = \frac{3}{2\omega_n} = \frac{3}{2 \times 3.53} = 0.424 \]

\[ \omega_n = \frac{\omega}{\sqrt{1 - \zeta^2}} = 3.53 \sqrt{1 - (0.424)^2} = 3.197 \]

\[ \pi = 3.197 \]

\[ T = 1.96 \text{ sec.} \]

Example 2.6. The open-loop transfer function of a servo system with unity feedback is given by

\[ G(S) = \frac{10}{(s + 2)(s + 5)} \]

Determine the damping ratio, undamped natural frequency of oscillation. What is the percentage overshoot of the response to a unit step input?

Solution: Given that

\[ G(S) = \frac{10}{(s + 2)(s + 5)} \]

\[ H(S) = 1 \]

\[ \text{characteristic equation} = 1 + G(S)H(S) = 0 \]

\[ 1 + \frac{10}{(s + 2)(s + 5)} = 0 \]

\[ \frac{10}{(s + 2)(s + 5)} \]
Example 2.6. Show that the system transfer function $Y(s)/X(s)$ has a zero in the right half $s$-plane. Obtain $y(t)$ when $x(t)$ is a unit step for the system shown in Fig. 2.20.

Solution: The transfer function is given by

$$\frac{Y(s)}{X(s)} = \frac{4}{s+2}$$

(Blocks in parallel)

$$\frac{Y(s)}{X(s)} = \frac{2(s-1)}{(s+1)(s+2)}$$

Fig. 2.20.

From the above equation, it is clear that there is a zero at $s = 1$ on the right half of the $s$-plane.

$$X(t) = 1$$

$$X(s) = \frac{1}{s}$$

$$Y(s) = \frac{6}{s+2} - \frac{4}{s+1} = 6 - \frac{4}{s+1}$$

Break the $\frac{4}{s+1}$ by partial fraction

$$Y(s) = 6 \left( \frac{1}{s+2} - \frac{1}{s+1} \right) - 4 \left( \frac{1}{s+1} \right) = \frac{3}{s+2} + \frac{4}{s+1}$$

Inverse Laplace of above $\frac{4}{s+1}$

$$\mathcal{L}^{-1} \left[ \frac{3}{s+2} + \frac{4}{s+1} \right] = e^{-2t} - 3e^{-t} + 4e^{-t}$$

Example 2.8. The open loop transfer function of a unity feedback system is given by

$$G(s) = \frac{K}{s(1+sT)}$$

where $K$ and $T$ are positive constants. By what factor should the amplifier gain be reduced so that the peak overshoot of unit step response of the system is reduced from 75% to 25%?

(KNIT Sultnagar, 1998-99)
Example 2.9. For given series RLC circuit, determine the undamped natural frequency and damping ratio of the circuit. Assume initial conditions are zero. \( R = 1 \text{K}, L = 10 \text{ mH}, C = 0.1 \text{mF} \). 

Solution: Find the transfer function and compare the characteristic equation with \( s^2 + 2\zeta\omega_n s + \omega_n^2 = 0 \) we get

\[
\zeta = \frac{R}{2\sqrt{L C}} = 0.5
\]

\[
\omega_n = \frac{1}{\sqrt{L C}} = 10^6 \text{ rad/sec}. \hspace{1cm} \text{Ans.}
\]

Example 2.10. Determine the value of \( K \) and \( H \) of the closed loop system so that the maximum overshoot in unit step response is 25\% and the peak time is 2 sec. Assume \( J \) = 1 kg m\(^2\).

Solution:

\[
\frac{C(S)}{R(S)} = \frac{K}{s^2 + KHs + K}
\]

Put \( J = 1 \)

\[
\frac{C(S)}{R(S)} = \frac{K}{s^2 + KHs + K}
\]

characteristic equation \( \lambda = \frac{-K}{2} \pm \sqrt{\frac{K^2}{4} - K} \).

\[
\omega_n = \sqrt{K}
\]

\[
2\zeta\omega_n = KH
\]

Given that \( M_p \) = 0.25 \( \Rightarrow \zeta = 0.25 \)

\[
\omega_n = 0.494
\]

Peak time

\[
t_p = \frac{\pi}{\omega_n\sqrt{1 - \zeta^2}}
\]

\[
\omega_n\sqrt{1 - \zeta^2} = \frac{\pi}{2} \Rightarrow \omega_n = 1.72 \text{ rad/sec}
\]

\[
\omega_n = \sqrt{K}
\]

\[
2\zeta\omega_n = KH
\]

\[
H = 0.471
\]

\[
K = 2.97 \text{ Nm}
\]

\[
\text{Ans.}
\]

Example 2.11. A second order control system is represented by the transfer function given below.

\[
\frac{Q_o(s)}{T(s)} = \frac{1}{s^2 + 6s + K}
\]

where \( Q_o(s) \) is the proportional output and \( T \) is the input torque. A step input \( 10 \text{ Nm} \) is applied to the system and test results are given below.

Example 2.12. A feedback system is described by the following transfer function

\[
G(s) = \frac{12}{s^2 + 4s + 16}, \quad H(s) = Ks
\]

The damping factor of the system is 0.8. Determine the overshoot of the system, and the value of \( K \).

Solution:

\[
\frac{C(s)}{R(s)} = \frac{G(s)}{1 + G(s)H(s)}
\]

\[
\frac{C(s)}{R(s)} = \frac{16}{s^2 + (4 + 16K)s + 16}
\]

\[
J = \frac{1}{0.8^2} = 1.128
\]

\[
f = 6.34
\]
Example 2.13. The open loop transfer function of a unity feedback control system is given by:

\[ G(s) = \frac{K}{s(1+sT)} \]

By what factor the amplifier gain \( K \) should be multiplied so that the damping ratio is increased from 0.3 to 0.9.

Solution:

The characteristic equation is

\[ s^2 + 2\zeta_1\omega_n s + \omega_n^2 = 0 \]

Compared with

\[ s^2 + 2\zeta_0\omega_n s + \omega_0^2 = 0 \]

which gives

\[ 2\zeta_0 = \frac{1}{T} \]

\[ \omega_0^2 = \frac{K}{T} \]

and

\[ \omega_n = \sqrt{\frac{K}{T}} \]

The damping ratio is

\[ \zeta_1 = 0.3; \quad \zeta_2 = 0.9 \]

Hence, the gain \( K_1 \) at which \( \zeta = 0.3 \) should be multiplied by \( 1/9 \) to increase the damping ratio from 0.3 to 0.9.

Example 2.14. The system shown in fig. 2.21a when subjected to a unit step input, the output response is shown in fig. 2.21b. Determine the value of \( K \) and \( T \) from the response curve.

![Response Curve Diagram](image)

The characteristic equation is

\[ s^2 + \frac{1}{T} s + \frac{K}{T} = 0 \]

Compare with

\[ s^2 + 2\zeta_0\omega_n s + \omega_n^2 = 0 \]

Then,

\[ \omega_n = \sqrt{\frac{K}{T}} \quad \text{and} \quad 2\zeta_0\omega_n = \frac{1}{T} \]

\[ M_p = 0.254 \quad \text{(given)} \]

\[ M_p = e^{\frac{-\pi}{\sqrt{1-\zeta^2}}} \times 100 \]

\[ 0.254 = e^{\frac{-\pi}{\sqrt{1-0.4^2}}} \quad \therefore \quad \zeta = 0.4 \]

\[ t_p = 3 \text{ sec. (given)} \]

\[ \omega_n = \sqrt{\frac{K}{T}} \quad \text{or} \quad T = \frac{1}{2\omega_n(1-0.4^2)} \]

\[ 2\zeta_0\omega_n = \frac{1}{T} \quad \text{or} \quad T = \frac{1}{2\zeta_0\omega_n} = \frac{1}{2 \times 0.4 \times 1.14} = 1.09 \]

\[ \omega_n = \sqrt{\frac{K}{T}} \quad \text{or} \quad \omega_n^2 = \frac{K}{T} \]

\[ K = 1.42 \]

Example 2.15. A system having a forward path transfer function \( G(S) = \frac{16}{s(s+1)} \) and unity feedback. Determine the value of undamped natural frequency, damping ratio. If tachometer feedback is introduced, the feedback path transfer function becomes \( (1+K_s) \). What should be the value of \( K \) to obtain damping ratio of 0.6. Also calculate the percentage peak overshoot, first undershoot, \( t_p \) and settling time within the 2\% of final value.
Solution: The characteristic equation with unity feedback

\[ 1 + \frac{\mathcal{G}(s)}{\mathcal{H}(s)} = 0 \quad \Rightarrow \quad \mathcal{H}(s) = 1 \]

\[ \frac{s^2 + \omega_n^2}{s(s+1)} = 0 \]

or compare with \( s^2 + 2\zeta\omega_n s + \omega_n^2 = 0 \)

\[ \omega_n^2 = 16 \]

\[ 2\zeta\omega_n = 1 \]

Now, consider \( \mathcal{H}(s) = 1 + Ks \)

\[ 1 + \frac{\mathcal{G}(s)}{\mathcal{H}(s)} = 0 \]

\[ \frac{1 + \frac{16}{s(s+1)}}{1 + Ks} = 0 \]

or, \( s^2 + (1 + 16K)s + 16 = 0 \)

\[ \omega_n^2 = 16 \]

\[ 2\zeta\omega_n = 1 + 16K \]

\[ 2 \times 6 \times 4 \times 1 + 16K \quad \therefore K = 0.2375 \]

\[ M_p = e^{-\zeta\zeta_1} \times 100 \]

When \( \zeta = 0.125 \), \( M_p = e^{-\zeta\zeta_1} \times 100 = 67.3\% \).

When \( \zeta = 0.6 \), \( M_p = e^{-\zeta\zeta_1} \times 100 = 9.48\% \).

First underdoot:

\[ t_p = e^{-\zeta\zeta_1} \times 100 = 0.009 \]

\[ \frac{\pi}{4\sqrt{1-(0.6)^2}} = 0.98 \text{ sec.} \]

\[ \frac{4}{\zeta\omega_n} \]

\[ \frac{4}{0.6} \times 4 = 8 \text{ sec.} \]

\[ \frac{4}{0.125 \times 4} = 8 \text{ sec.} \]

\[ \frac{4}{0.6 \times 4} = 1.67 \text{ sec.} \]

Example 2.16: In example 2.13, by what factor the time constant 'T' should be multiplied so that the damping ratio is reduced from 0.8 to 0.2.

Example 2.17: Consider the system shown in fig. 2.22. Determine the value of \( k \) such that the system damping ratio \( \zeta \) is 0.5. Then obtain the rise time \( (t_r) \), peak time \( (t_p) \), maximum overshoot \( M_p \), and settling time \( (t_s) \) in the unit step response.

\[ \frac{c(s)}{R(s)} \]

\[ \frac{c(s)}{R(s)} = \frac{s}{s^2 + (0.8 + 16k)s + 16} \]

\[ \omega_n^2 = 16 \quad \therefore \omega_n = 4 \text{ rad/sec.} \]

\[ 2\zeta\omega_n = 0.8 + 16k \]

\[ 2 \times 0.5 \times 4 = 0.8 + 16k \quad \therefore k = 0.2 \]

\[ \frac{\pi - \tan^{-1} \sqrt{1 - \frac{\zeta^2}{\zeta^2}}} {\omega_n \sqrt{1 - \frac{\zeta^2}{\zeta^2}}} = \frac{\pi - \tan^{-1} \sqrt{1 - 0.5^2}} {4\sqrt{1 - 0.5^2}} = 0.605 \text{ sec.} \]

\[ \frac{\pi}{4\sqrt{1 - 0.5^2}} = 0.906 \text{ sec.} \]

\[ M_p = e^{\frac{k}{\zeta}} \times 100 = e^{\frac{0.2}{0.5}} \times 100 = 163\% \]

\[ \frac{4}{0.5 \times 4} = 2 \text{ sec.} \quad \text{ Ans.} \]
Example 2.18. A system shown in fig. 2.23, determine the value of $K$ and $k$ such that the system has a damping ratio $\zeta$ of 0.7 and an undamped natural frequency $\omega_n$ of 4 rad/sec.

Solution: Find $\frac{c(s)}{R(s)}$ by block reduction method.

\[
\frac{c(s)}{R(s)} = \frac{K}{s^2 + (2 + 2K)s + K} = \frac{K}{s^2 + 2\zeta\omega_n s + \omega_n^2} = \frac{K}{s^2 + 2\zeta\omega_n s + \omega_n^2}
\]

characteristic equation $s^2 + (2 + 2K)s + K = 0$ compare with $s^2 + 2\zeta\omega_n s + \omega_n^2 = 0$

$\omega_n^2 = K$ \quad $K = 16$ \quad $K = 16$

$2\zeta\omega_n = 2 + 2K$ \quad $K = 16$

$2 \times 0.7 \times 4 = 2 + 16k$ \quad $k = 0.225$

Example 2.19. A thermometer requires 1 min. to indicate 98% of the response to a step input. Assuming the thermometer to be a first order system, find the time constant.

Solution: For first order system

$\frac{c(t)}{t} = 1 - e^{-t/T}$

$T = \text{time constant} = 7$

$c(t) = 0.98 \quad 0.98 = 1 - e^{-t/T}$ \quad $T = 15.33$ sec.

Time constant = 15.33 sec.

Example 2.20. Consider the fig. 2.24. The damping ratio of the system is 0.158 and the undamped natural frequency is 3.16 rad/sec. To improve the relative stability (fig. 2.24b) we employ tachometer feedback. Determine the value of $K_b$ so that the damping ratio of the system is 0.5.

Solution: From fig. 2.24 (b) we can determine $\frac{c(s)}{R(s)}$ by block reduction method.

\[
\frac{c(s)}{R(s)} = \frac{10}{s(s + 10)}
\]

characteristic equation $s^2 + (1 + 10K_b)s + 10 = 0$ compare with $s^2 + 2\zeta\omega_n s + \omega_n^2 = 0$

$\omega_n^2 = 1 + 10K_b$

$\omega_n^2 = 3.16 \text{ rad/sec.}$

$K_b = 0.216$ \quad $K_b = 0.216$

Example 2.21. Obtain the unit step response of a unity feedback system whose open loop transfer function is

\[
G(s) = \frac{4}{s(s + 5)}
\]

Solution:

\[
\frac{c(s)}{R(s)} = \frac{G(s)}{1 + G(s)H(s)} = \frac{4}{s(s + 5) + 1}
\]

\[
\frac{c(s)}{R(s)} = \frac{4}{s^2 + 5s + 4}
\]

\[
\frac{c(s)}{R(s)} = \frac{4}{s^2 + 5s + 4}
\]

\[
\frac{c(s)}{R(s)} = \frac{4}{s^2 + 5s + 4}
\]

Given

\[
R(s) = \frac{1}{s}
\]

Break the equation by partial fraction

\[
\frac{4}{s(s + 1)(s + 4)} = \frac{A}{s + 1} + \frac{B}{s + 4} + \frac{C}{s + 4}
\]

$A = 1, B = -\frac{4}{3}, C = \frac{1}{3}$

\[
\frac{c(s)}{R(s)} = \frac{1}{s} + \frac{4}{3(s + 1)} + \frac{1}{3(s + 4)}
\]

Invers Laplace of above eqn.

\[
\mathcal{L}^{-1}\{c(s)\} = \mathcal{L}^{-1}\left\{\frac{1}{s}\right\} - \frac{4}{3}\left\{\frac{1}{s + 1}\right\} + \frac{1}{3}\left\{\frac{1}{s + 4}\right\}
\]

\[
c(t) = 1 - \frac{4}{3} e^{-t} - \frac{1}{3} e^{-4t}
\]

Ans.

Example 2.22. Consider the unit step response of a unity feedback control system whose open loop transfer function is

\[
G(s) = \frac{1}{s(s + 1)}
\]

Obtain the rise time, peak time, maximum overshoot and settling time.

Solution: Characteristic eq $1 + G(s)H(s) = 0$

\[
1 + \frac{1}{s(s + 1)} - 1 = 0
\]

\[
s^2 + 1 = 0
\]

Compare with $s^2 + 2\zeta\omega_n s + \omega_n^2 = 0$

$\omega_n^2 = 1$

$\omega_n = 1 \text{ rad/sec.}$

$\zeta = 0.5$
Example 2.23. Consider the closed loop system given by

\[
\frac{c(s)}{R(s)} = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}
\]

Determine the values of \(\zeta\) and \(\omega_n\) so that the system responds to a step input with approximately 5% overshoot and with a settling time of 2 sec.

Solution: Given that

\[
t_s = 2 \text{ sec}, \\
M_p = 5\% = 0.05 \\
M_p = e^{-\frac{\pi}{2(1-\zeta^2)}}
\]

\[
0.05 = e^{-\frac{\pi}{2(1-\zeta^2)}} \\
\zeta = 0.689
\]

\[
t_s = \frac{4}{\zeta\omega_n} \text{ or, } \omega_n = \frac{4}{t_s\zeta}
\]

\[
\omega_n = \frac{4}{0.689 \times 2} = 2.9 \text{ rad/sec.} \\
\omega_n = 2.9 \text{ rad/sec.}
\]

Example 2.24. Fig. 2.25 is a block diagram of a space-vehicle attitude control system. Assuming the time constant \(T\) of the controller to be 3 sec and the ratio of torque to inertia \(K/J\) to be \(\frac{2}{9}\) rad/sec sec\(^2\) find the damping ratio of the system.

Solution: The characteristic eqn \(1 + G(s) H(s) = 0\)

\[
1 + \frac{K(Ts + 1)}{J^2} = 0
\]

\[
\frac{K}{J^2} = \frac{1}{1 + sT} = C(s)
\]

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\[
\begin{align*}
\text{or,} & \quad s^2 + 3Ks + K = 0 \\
\text{or,} & \quad s^2 + 3\frac{K}{T}s + \frac{K}{T} = 0 \\
\end{align*}
\]

\[
\text{compare with } \quad s^2 + 2\zeta\omega_n s + \omega_n^2 = 0 \\
\omega_n^2 = \frac{K}{T} \\
2\zeta\omega_n = 3\frac{K}{T} \\
\omega_n^2 = \frac{2}{9}
\]

\[
\omega_n = 0.471 \text{ rad/sec}
\]

\[
2\times\zeta\times0.471 = 3 - \frac{2}{9} = \zeta = 0.707
\]

\[
\zeta = 0.707 \quad \text{Ans.}
\]

Example 2.25. Measurement conducted on a servomechanism show the system response to be

\[
c(t) = 1 + 0.2 e^{-0.1t} - 1.2 e^{-10t}
\]

when subjected to a unit step input.

(a) Obtain the expression for the closed loop transfer function.

(b) Determine the undamped natural frequency and damping ratio of the system.

Solution:

\[
c(s) = \frac{1}{s^2 + 0.2\frac{s}{60} + \frac{1.2}{s+10}}
\]

\[
c(s) = \frac{600}{s(s+10)(s+60)}
\]

\[
\zeta = 0.689 \\
\omega_n = 2.9 \text{ rad/sec.}
\]

\[\text{(a) Transfer function } = \frac{c(s)}{R(s)} = \frac{s(s+10)(s+60)}{1/s} = \frac{600}{s(s+10)(s+60)}
\]

\[\text{Characterisic equation } = s^2 + 70s + 600, \text{ compare with } s^2 + 2\zeta\omega_n s + \omega_n^2 = 0
\]

\[
\omega_n = 24.49 \text{ rad/sec.}
\]

\[
\zeta = 1.42 \quad \text{Ans.}
\]

2.9 EFFECT OF FEEDBACK ON TIME CONSTANT OF A CONTROL SYSTEM

Consider the open loop transfer function as shown in fig. 2.26.

\[
\begin{align*}
R(s) & \quad \frac{1}{1+sT} \quad C(s)
\end{align*}
\]
Thus for positive values of $M$, the time constant $\frac{T}{1+M}$ is less than time constant $T$. It means that the time constant of closed loop system is less than the time constant of open loop system. Less time constant means response is faster. Therefore feedback improves the time response of the system.

### SUMMARY

Time response consists of two parts (a) transient response, (b) steady state response. Transient response is the part of the response which goes to zero as time increases and steady state response is the part of the total response after transient has died.

**Test Input signals:**

1. Step function is also known as displacement function. Laplace transform is $\frac{1}{s}$.
2. Ramp function: Ramp function starts from origin and increase or decreases linearly with time. Laplace transform is $\frac{1}{s^2}$.
3. Parabolic function: Laplace transform is $\frac{1}{s^3}$.
4. Impulse function: Impulse function has zero value everywhere except at $t=0$ where the amplitude is infinite.
   
   - Step function is the derivative of ramp function.
   
   $\frac{d}{dt} \left( \text{unit ramp function} \right) = s \frac{1}{s^2} = \frac{1}{s} = \text{step function}$

   - Impulse function is the derivative of step function
   
   $\frac{d}{dt} \left( \text{unit step function} \right) = s \frac{1}{s} = 1$

   - Ramp function is the derivative of parabolic function
   
   $\frac{d}{dt} \left( \text{parabolic function} \right) = \frac{s}{s^2} = \frac{1}{s} = \text{ramp function}$

   Transient response depends upon:
   
   (a) Location of closed loop poles in s-plane
   
   (b) Order of the system

   Response of first order system with unit step input is given by
   
   $x(t) = 1 - e^{-t/T}$

   Response of first order system with unit ramp input is given by
   
   $x(t) = t - T + \frac{t^2}{2T}$

   Response of first order system with unit impulse input is given by
   
   $x(t) = \frac{1}{T} e^{-t/T}$

   The closed loop transfer function of second order system is
\[
\frac{C(t)}{R(t)} = \frac{\frac{\omega_n^2}{s^2 + 2\zeta \omega_n s + \omega_n^2}}{s^2 + 2\zeta \omega_n s + \omega_n^2}
\]

Where \( \omega_n \) = natural frequency of oscillations
\( \zeta \) = damping ratio

Response of second order system with unit step input is given by

\[
C(t) = 1 - \frac{\omega_n}{\sqrt{1-\zeta^2}} \sin \left[ \frac{\omega_n \sqrt{1-\zeta^2}}{\zeta} \right] \frac{t}{\tan^{-1} \sqrt{1-\zeta^2}}
\]

The system is underdamped for \( 0 < \zeta < 1 \)
The system is critical damped for \( \zeta = 1 \)
The system is overdamped for \( \zeta > 1 \)
The system is undamped for \( \zeta = 0 \)

**Delay time** \((t_d)\): The delay time is the time required for the response to reach 50% of the final value in first time.

**Rise time** \((t_r)\): It is time required for the response to rise from 10% to 90% of its final value for overdamped systems and 0 to 100% for underdamped systems.

**Peak time** : The peak time is time required for the response to reach the first peak of the time response of first peak overshoot.

**Maximum overshoot** : It is the normalized difference between the peak of the time response and steady output.

**Setting time** \((t_s)\): Setting time is time required for the response to reach and stay within a specified range (2% to 5%) of its final value.

**Steady state error** : Steady state error is the difference between actual output and desired output as time \( t \) tends to infinity.

\[
e_{ss} = \lim_{t \to \infty} [r(t) - o(t)]
\]

Rise time is given by

\[
t_r = \frac{\pi}{\omega_n} \sqrt{1-\zeta^2}
\]

Peak time is given by

\[
t_p = \frac{2\pi}{\omega_n} \sqrt{1-\zeta^2}
\]

Maximum overshoot is given by

\[
\% M_p = e^{-\frac{\sqrt{1-\zeta^2}}{\zeta}} \times 100
\]

Settling time is given by

\[
t_s = \frac{4}{\zeta \omega_n}
\]

Response of second order system with unit impulse is given by

\[
c(t) = \frac{\omega_n}{\sqrt{1-\zeta^2}} e^{-\omega_n \sqrt{1-\zeta^2} t} \sin \left( \omega_n \sqrt{1-\zeta^2} t + \frac{\pi}{2} \right)
\]

Response of second order system with unit ramp input

\[
c(t) = t - \frac{\omega_n}{\sqrt{1-\zeta^2}} e^{-\omega_n \sqrt{1-\zeta^2} t} \left[ \omega_n \sqrt{1-\zeta^2} t + \frac{\pi}{2} \right]
\]

**EXERCISE**

1. A unity feedback system is characterized by an open loop transfer function

\[
G(s) = \frac{K}{s(s + 20)}
\]

Determine the gain \( K \) so that the system will have a damping ratio of 0.6. For this value of \( K \) calculate settling time, peak overshoot and time to peak overshoot for a unit step input.

2. A feedback system is described by the following transfer function

\[
G(s) = \frac{20}{s^2 + 4s + 25}, \quad H(s) = Ks
\]

The damping factor of the system is 0.8. Determine the overshoot of the system.

3. The forward path transfer function of a system is given by

\[
G(s) = \frac{4}{s(s + 6)}
\]

Obtain an expression for unit step response of the system. Assume unity feedback system.

4. The open loop transfer function of a servosystem with unity feedback system is given by

\[
G(s) = \frac{K}{s(s + 2)(s + 6)}
\]

Determine the damping ratio, undamped natural frequency, maximum overshoot for unit step input and \( K = 10 \).

5. The open loop transfer function of a unity feedback system is \( G(s) = \frac{K}{s(1 + sT)} \). By what factor the gain \( K \) be reduced so that the overshoot is reduced from 60% to 20%.

6. The block diagram of a position control system with velocity feedback is shown in fig. 2.28. Determine the value of \( v \) so that the step response has a maximum overshoot of 20%.

[Diagram]

7. For the closed loop system given by

\[
\frac{C(s)}{R(s)} = \frac{\frac{w_n^2}{s^2 + 2\zeta w_n s + \omega_n^2}}{s^2 + 2\zeta w_n s + \omega_n^2}
\]

Determine the values of \( \zeta \) and \( w_n \) so that the system responds to a unit step input with 10% overshoot and with settling time of 4 sec.
8. For a unity feedback system with \( G(s) = \frac{4}{s(s + 2)} \) obtain an expression for unit step response of the system.

9. Determine the time response \( y(t) \) for
   \[ Y(s) = \frac{4}{s(s + 1)(s + 2)} \]
   \[ Y(s) = \frac{4}{s(s - 1)(s - 2)} \]

10. What are the time domain quantities which characterise a transient response? Derive an expression for percentage overshoot of a 2nd order system. (A.M.L. Aligarh, B.E. 1982-83)

11. The open loop transfer function of a unity feedback system is
   \[ G(s) = \frac{K}{s + 1} \]
   Find the steady state error for an input of a unity feedback system as
   \[ r(t) = 1 + \frac{t^2}{2} \]  
   (A.M.L. Aligarh, B.E. 1986-87)

**Semi-Objective Type Questions**

(i) Define transient response
(ii) Sketch various test input signals for transient analysis.
(iii) Write mathematical expressions for various test input signals.
(iv) Derive the expression for response of the first order system with unit step input.

(v) Define delay time, rise time and peak time.
(vi) Derive the expression for maximum overshoot \( M_p \)
(vii) For a system having transfer function
   \[ G(s) = \frac{64}{s^2 + 5s + 64} \]
   determine: (i) \( \omega_n \), (ii) \( \zeta \), (iii) \( \omega_d \)
   (Ans. \( \omega_n = 8 \text{ rad/sec} \), \( \zeta = 0.3125 \), \( \omega_d = 7.6 \text{ rad/sec} \))

(viii) Draw a typical transient response output for an underdamped 2nd order system and show the locations of all the transient response specifications.
(ix) For a first order system find out the output of the system when the input applied to the system is a unit ramp.

(x) What is steady state error?
(xi) What is the effect of feedback on time constant of a control system?
(xii) Explain the relationship between damping ratio and % overshoot.
(xiii) What are time domain specifications?

---

**Chapter 3**

**Error Analysis**

3.1. **CLASSIFICATION OF CONTROL SYSTEM**

Consider the open loop transfer function

\[ G(s)H(s) = \frac{K(s + s_1)(s + s_2)}{s^n(s + s_{n-1})(s + s_n)} \]  
... (3.1)

In equation (3.1), the poles are at \( s = -\frac{1}{T_p}, s = -\frac{1}{T_z} \), and zeros are at \( s = -\frac{1}{T_H} \).

The equation having a term \( s^n \) in denominator, \( m \) is the number of poles at the origin.

If \( m = 1 \), it means the system has a pole at origin of the \( s \)-plane and is said to type I system.

A system having no pole at origin of the \( s \)-plane, is said to be type II (zero) system i.e., \( m = 0 \).

3.2. **STEADY STATE ERROR**

The steady state error is the difference between the input and output of the system during steady state. For accuracy the steady state error should be minimum.

Consider a closed loop control system shown in Fig. 3.1.

\[ E(s) = \frac{1}{1 + G(s)H(s)} \]  
... (3.2)

or,

\[ E(s) = \frac{R(s)}{1 + G(s)H(s)} \]  
... (3.2)

The steady state error of the system is obtained by applying final value theorem.

\[ e_{ss} = \lim_{s \to 0} sE(s) \]  
... (3.3)

\[ e_{ss} = \lim_{s \to -\infty} sE(s) \]  
... (3.4)

For unity feedback system \( H(s) = 1 \)

\[ e_{ss} = \lim_{s \to -\infty} \frac{R(s)}{1 + G(s)} \]  
... (3.5)

From the equation (3.4) or (3.5) it is clear that the steady state error depends on the input and open loop transfer function.

3.3. **STATIC ERROR COEFFICIENTS**

(a) Static-Position Error Constant (or Coefficient) \( K_p \)

The steady state error is given by equation 3.4.
\[
\varepsilon_s = \lim_{s \to 0} \frac{R(s)}{1 + G(s)H(s)}
\]

For unit step input \( R(s) = \frac{1}{s} \), the steady state error is given by

\[
\varepsilon_s = \lim_{s \to 0} \frac{1}{1 + G(s)H(s)} = \lim_{s \to 0} \frac{1}{1 + \frac{1}{K_p}} = \frac{1}{K_p}
\]

\( K_p \) = static position error constant = \( \lim_{s \to 0} G(s)H(s) \)

\( \text{(b) Static Velocity Error Constant (or Coefficient) } K_v \)

\[
\varepsilon_v = \lim_{s \to 0} R(s) = \frac{1}{s^2 (1 + G(s)H(s))}
\]

Steady state error with a unit ramp input is given by \( R(s) = \frac{1}{s^2} \)

\[
\varepsilon_v = \lim_{s \to 0} \frac{1}{s^2} \frac{1}{1 + G(s)H(s)} = \lim_{s \to 0} \frac{1}{s G(s)H(s)} = \frac{1}{K_v}
\]

\( \text{Where } K_v = \lim_{s \to 0} G(s)H(s) \) = static velocity error coeff.

\( \text{(c) Static Acceleration Error Constant } K_a \)

The steady state error of the system with unit parabolic input is given by

\[
R(s) = \frac{1}{s^3}
\]

\[
\varepsilon_a = \lim_{s \to 0} \frac{1}{s^3} \frac{1}{1 + G(s)H(s)} = \lim_{s \to 0} \frac{1}{s^4 G(s)H(s)} = \frac{1}{K_a}
\]

\( \text{Where } K_a = \lim_{s \to 0} s^2 G(s)H(s) \) = static acceleration constant.

3.4. STEADY STATE ERROR FOR DIFFERENT TYPE OF SYSTEMS

\( \text{(a) Type Zero System with Unit Step Input} \)

\[
G(s)H(s) = \frac{K(1+sT_1)(1+sT_2)...}{(1+sT_a)(1+sT_b)...
\]

From eq. (3.1)

\[
K_p = \lim_{s \to 0} G(s)H(s) = \lim_{s \to 0} \frac{K(1+sT_1)(1+sT_2)...}{(1+sT_a)(1+sT_b)...} = K
\]

\[
\varepsilon_s = \frac{1}{1 + K_p} = \frac{1}{1 + K}
\]

\( \varepsilon_s = \frac{1}{1 + K} \)

Hence, for type zero system the static position error constant \( K_p \) is finite.

\( \text{(b) Type '0' System with Unit Ramp Input} \)

\[
K_p = \lim_{s \to 0} s \cdot G(s)H(s) = \lim_{s \to 0} \frac{K(1+sT_1)(1+sT_2)...}{(1+sT_a)(1+sT_b)...} = 0
\]

\[
\varepsilon_v = \frac{1}{K_v} = \infty
\]

\( \varepsilon_v = \infty \)

\( \text{For type '0' system, the steady state error is infinite for ramp and parabolic inputs. Hence, the} \)

ramp and parabolic inputs are not acceptable.

\( \text{2(a) Type 'I' System with Unit Step Input} \) \( m = 1 \)

\[
G(s)H(s) = \frac{K(1+sT_1)(1+sT_2)...}{s(1+sT_a)(1+sT_b)...}
\]

\[
K_p = \lim_{s \to 0} G(s)H(s) = \lim_{s \to 0} \frac{K(1+sT_1)(1+sT_2)...}{s(1+sT_a)(1+sT_b)...} = \infty
\]

\[
\varepsilon_s = \frac{1}{1 + K_p} = 0 \quad \varepsilon_s = 0
\]

\( \text{For type 'I' system, with unit parabolic input} \)

\[
K_p = \lim_{s \to 0} s \cdot G(s)H(s) = \lim_{s \to 0} \frac{K(1+sT_1)(1+sT_2)...}{s(1+sT_a)(1+sT_b)...} = \infty
\]

\[
\varepsilon_v = \frac{1}{K_v} = 0 \quad \varepsilon_v = 0
\]

\( \text{For type 'I' system, with unit parabolic input} \)

\[
K_a = \lim_{s \to 0} s^2 G(s)H(s)
\]

\[
\varepsilon_a = \frac{1}{K_a} = \frac{1}{K}
\]

\( \text{For type 'I' system, it is clear that for type 'I' system step input and ramp inputs are acceptable and } \)

parabolic input is not acceptable.
3. (a) Type '2' system with unit step input

\[ G(s)H(s) = \frac{K(1+sT_1)(1+sT_2)}{s^2(1+sT_1)(1+sT_2)} \]

\[ K_p = \lim_{s \to 0} G(s)H(s) = \lim_{s \to 0} \frac{K(1+sT_1)(1+sT_2)}{s^2(1+sT_1)(1+sT_2)} = \infty \]

\[ e_s = \frac{1}{1+K_p} = 0 \quad e_n = 0 \]

(b) Type '2' system with unit ramp input

\[ K_v = \lim_{s \to 0} sG(s)H(s) \]

\[ = \lim_{s \to 0} sK(1+sT_1)(1+sT_2) \]

\[ = \lim_{s \to 0} \frac{sK(1+sT_1)(1+sT_2)}{s^2(1+sT_1)(1+sT_2)} = \infty \]

\[ e_s = \frac{1}{K_v} = 0 \quad e_n = 0 \]

(c) Type '2' system with unit parabolic input

\[ K_n = \lim_{s \to 0} s^2G(s)H(s) \]

\[ = \lim_{s \to 0} s^2K(1+sT_1)(1+sT_2) \]

\[ = \lim_{s \to 0} \frac{s^2K(1+sT_1)(1+sT_2)}{s^2(1+sT_1)(1+sT_2)} = K \]

\[ e_s = \frac{1}{K_n} = \frac{1}{K} \quad e_n = \frac{1}{K} \]

Hence, for type '2' system all three inputs (step, ramp and parabolic) are acceptable. From table 3.1, the diagonal elements are the finite values of steady state error.

<table>
<thead>
<tr>
<th>Type '0' system</th>
<th>Type '1' system</th>
<th>Type '2' system</th>
</tr>
</thead>
<tbody>
<tr>
<td>UNIT STEP INPUT</td>
<td>( \frac{1}{1+K} )</td>
<td>0</td>
</tr>
<tr>
<td>UNIT RAMP INPUT</td>
<td>( \infty )</td>
<td>( \frac{1}{K} )</td>
</tr>
<tr>
<td>UNIT PARABOLIC INPUT</td>
<td>( \infty )</td>
<td>( \infty )</td>
</tr>
</tbody>
</table>

Example 3.1. The open loop transfer function of unity feedback system is given by

\[ G(s) = \frac{50}{(1+0.1s)(s+10)} \]

Determine the static error coefficients \( K_p \), \( K_v \), and \( K_n \).

Solution:

\[ K_p = \lim_{s \to 0} G(s)H(s) = \lim_{s \to 0} s \frac{50}{(1+0.1s)(s+10)} = \infty \]

\[ K_v = \lim_{s \to 0} sG(s)H(s) = \lim_{s \to 0} s \frac{50}{s(1+0.1s)(s+10)} = 0 \]

\[ K_n = \lim_{s \to 0} s^2G(s)H(s) = \lim_{s \to 0} s^2 \frac{50}{s^2(1+0.1s)(s+10)} = 0 \]

Example 3.2. The forward path transfer function of a unity feedback control system is given by

\[ G(s) = \frac{5(s^2 + 2s + 100)}{s^2(s+5)(s^2 + 3s + 10)} \]

Determine the step, ramp and parabolic error coefficients. Also determine the type of the system.

Solution:

\[ K_p = \lim_{s \to 0} G(s)H(s) = \lim_{s \to 0} s \frac{5(s^2 + 2s + 100)}{s(s+5)(s^2 + 3s + 10)} = \infty \]

\[ K_v = \lim_{s \to 0} sG(s)H(s) = \lim_{s \to 0} s^2 \frac{5(s^2 + 2s + 100)}{s^2(s+5)(s^2 + 3s + 10)} = \infty \]

\[ K_n = \lim_{s \to 0} s^2G(s)H(s) = \lim_{s \to 0} s^3 \frac{5(s^2 + 2s + 100)}{s^3(s+5)(s^2 + 3s + 10)} = 10 \]

In denominator the value of \( m = 2 \). Hence, the given system is type '2' system.

Example 3.3. The block diagram of an electronic pacemaker is given in fig. 3.2. Determine the steady state error for unit ramp input when \( K = 400 \) Also, determine the value of \( K \) for which the steady state error to a unit ramp will be 0.02.

Solution: Given that \( K = 400 \)

\[ R(s) = \frac{1}{s} \]

\[ H(s) = \frac{1}{s+20} \]

\[ G(s)H(s) = \frac{K}{s(s+20)} \]

Steady state error is given by \( e_s = \lim_{s \to 0} s \frac{1}{1+G(s)H(s)} \)

\[ e_s = \lim_{s \to 0} s \frac{1}{s+20} = \frac{1}{s+20} \frac{1}{K} = \frac{s+20}{K} = \frac{400}{K} = 0.05 \quad \text{Ans.} \]

Now \( e_s = 0.02 \) (given)

\[ e_s = \lim_{s \to 0} s \frac{1}{s+20} = \frac{1}{s+20} \frac{1}{K} = \frac{s+20}{K} = \frac{20}{K} \]

\[ K = 1000 \quad \text{Ans.} \]
Example 3.4. For a unity feedback control system the forward path transfer function is given by

\[ G(s) = \frac{20}{s(s + 2)(s^2 + 2s + 20)} \]

Determine the steady state error of the system. When the inputs are (i) 5 (ii) 5t (iii) \( \frac{3t^2}{2} \).

Solution:
(i) \( r(t) = 5 \) : \( R(s) = 5 \)

\[ e_{ss} = \lim_{s \to 0} s \cdot E(s) = \lim_{s \to 0} \frac{r(s)}{1 + G(s)H(s)} = \lim_{s \to 0} \frac{5s(s + 2)(s^2 + 2s + 20)}{s(s + 2)(s^2 + 2s + 20)} = 0 \]

(ii) \( R(s) = \frac{5}{s^2} \)

\[ e_{ss} = \lim_{s \to 0} s \cdot E(s) = \lim_{s \to 0} \frac{r(s)}{1 + G(s)H(s)} = \lim_{s \to 0} \frac{5s(s + 2)(s^2 + 2s + 20)}{s(s + 2)(s^2 + 2s + 20) + 20} = 10 \]

(iii) \( R(s) = \frac{3}{s} \)

\[ e_{ss} = \lim_{s \to 0} s \cdot E(s) = \lim_{s \to 0} \frac{r(s)}{1 + G(s)H(s)} = \lim_{s \to 0} \frac{3s(s + 2)(s^2 + 2s + 20) + 20}{s(s + 2)(s^2 + 2s + 20)} = \infty \]

Example 3.5. The open loop transfer function of a unity feedback system is given by

\[ G(s) = \frac{108}{s(s + 4)(s^2 + 3s + 12)} \]

Find the static error coefficients and steady state error of the system when subjected to an input given by \( r(t) = 2 + 5t + 2t^2 \).

Solution:
\[ K_p = \lim_{s \to 0} G(s) = \lim_{s \to 0} \frac{108}{s(s + 4)(s^2 + 3s + 12)} = \infty \]

\[ K_0 = \lim_{s \to 0} s \cdot G(s) = \lim_{s \to 0} \frac{108}{s^2(s + 4)(s^2 + 3s + 12)} = \infty \]

\[ K_r = \lim_{s \to 0} s^2 G(s) = \lim_{s \to 0} \frac{108}{s^2(s + 4)(s^2 + 3s + 12)} = \frac{108}{48} \]

\( r(t) = 2 + 5t + 2t^2 \)

\( \beta(s) = \frac{2}{s^2} + \frac{5}{s} + \frac{4}{3} \)

Example 3.6. Determine the type of the following unity feedback systems for which the forward path transfer function is given:
(a) \( G(s) = \frac{K}{s(s + 10)(s + 5)} \)
(b) \( G(s) = \frac{20}{s^2(s + 2)} \)
(c) \( G(s) = \frac{K}{s^2(s + 5)(s + 2)} \)
(d) \( G(s) = \frac{K}{s(s + 1)(s + 2)} \)

Solution:
(a) Since in denominator the power of \( s \) is zero i.e. \( m = 0 \) hence it is type zero.
(b) type one system
(c) \( m = 3 \), type '3' system
(d) \( m = 1 \), type '1' system

Example 3.7. A servomechanism is designed to keep a radar antenna pointed at a flying aeroplane. If the aeroplane is flying with a velocity of 600 Km/hr, at a range of 2 km. and the maximum tracking error is to be within 0.1', determine the required velocity error coefficient. (GATE-1994)

Solution: The block diag. of servo mechanism is shown in fig. 3.3.

Angular velocity = \( \frac{600}{2} = 300 \) rad/hr.

Angular velocity error = \( \frac{300}{3600} = 0.083 \) rad/sec.

\[ R(s) = 0.083 \]

\[ \beta(s) = \frac{1}{s^2} \]

\[ \frac{1}{1 + G(s)H(s)} = \frac{1}{1 + \frac{1}{s^2}} = \frac{1}{s^2} \]

\[ e = \lim_{s \to 0} s \cdot \beta(s) = \lim_{s \to 0} \frac{s(1 + s)}{s(1 + s) + K_p} = 0.083 \frac{1}{K_p} \]

\[ 0.083 = 0.083 \frac{1}{K_p} \]

\[ K_p = 0.83 \text{ per degree} \]

Required velocity error coefficient \( K_p = 0.83 \) per degree

Example 3.8. For the system shown in fig. 3.4. Determine \( K_p \) and \( e_{ss} \) for unit step input.

Solution: \( K_p = \lim_{s \to 0} G(s) \cdot H(s) = \lim_{s \to 0} \frac{1}{s^2(s + 4)(s + 2)} = \frac{1}{2} \)

\[ e_{ss} = \frac{1}{1 + K_p} = \frac{1}{1 + \frac{1}{2}} = \frac{1}{3/2} = \frac{2}{3} \]

\( K_p = 1/2 \)

\( e_{ss} = 2/3 \) \( \text{Ans.} \)

Fig. 3.3.
Example 3.9. For a given system shown in fig. 3.5, determine the actuating signal $E_a(s)$. Also determine the position error constant for unit step input.

Solution:

$$E_a(s) = 20R(s) - B(s)$$

$$C(s) = \frac{20E_a(s)}{s(1 + 0.05s)}$$

$$B(s) = C(s) \frac{1 + 0.01s}{1 + 0.05s}$$

$$E(s) = \frac{20s(1 + 0.01s)E_a(s)}{s(1 + 0.05s)^2}$$

$$E_a(s) = 1 + \frac{20s(1 + 0.01s)^2}{s(1 + 0.05s)^2}$$

$$E_a(s) = \frac{20R(s)}{1 + 20(1 + 0.05s)^2}$$

For step input:

$$R(s) = \frac{1}{s}$$

$$E_a(s) = \frac{20(1 + 0.05s)}{s(1 + 0.05s)^2 + 20(1 + 0.01s)}$$

$$\epsilon_s = \lim_{s \to 0} s E(s) = \lim_{s \to 0} s \frac{20(1 + 0.05s)^2}{s(1 + 0.05s)^2 + 20(1 + 0.01s)} = 0$$

But

$$K_p = \frac{a}{b}$$

Ans.

Example 3.10. Consider a unity feedback control system with the closed loop transfer function

$$\frac{C(s)}{R(s)} = \frac{Ks + b}{s^2 + as + b}$$

Determine the open loop transfer function. Show that the steady state error in the unit ramp input response is given by

$$\epsilon_s = \frac{a - K}{b}$$

Solution: For closed loop control system

$$\frac{C(s)}{R(s)} = \frac{G(s)}{1 + G(s)H(s)}$$

$$G(s) = \frac{Ks + b}{s^2 + as + b}$$

Open loop transfer function $G(s) = \frac{Ks + b}{s^2 + s(a - K)}$

Steady state error $\epsilon_s = \lim_{s \to 0} s R(s)$

$$\epsilon_s = \lim_{s \to 0} s \frac{1}{\frac{1}{1 + G(s)H(s)}}$$

$$\epsilon_s = \lim_{s \to 0} s \frac{1}{\frac{1}{1 + \frac{Ks + b}{s^2 + s(a - K)}}} = \lim_{s \to 0} s \frac{1}{\frac{1}{s^2 + s(a - K) + Ks + b}}$$

$$\epsilon_s = \frac{a - K}{b}$$

Proved.

3.5. Dynamic Error Coefficients

For a given system, the static error coefficients give the limited information. The error function is given by

$$E(s) = \frac{1}{1 + C(s)H(s)}$$

For unity feedback system

$$\frac{E(s)}{R(s)} = \frac{1}{1 + G(s)}$$

The equation 3.11 can be expressed in polynomial form (ascending power of $s$)

$$E(s) = \frac{1}{K_1 + \frac{1}{K_2} s + \frac{1}{K_3} s^2 + \ldots}$$

or,

$$E(s) = \frac{1}{K_1} R(s) + \frac{1}{K_2} s R(s) + \frac{1}{K_3} s^2 R(s) + \ldots$$

Take inverse laplace of eqn. 3.13, the error given by

$$\epsilon(t) = \frac{1}{K_1} r(t) + \frac{1}{K_2} s r(t) + \frac{1}{K_3} s^2 r(t) + \ldots$$

Steady state error is given by

$$\epsilon_s = \lim_{s \to 0} s E(s)$$

Now, suppose if the input is unit step i.e. $R(s) = \frac{1}{s}$ then from 3.13

$$\epsilon_s = \lim_{s \to 0} s \left[ \frac{1}{K_1} + \frac{1}{K_2} + \frac{1}{K_3} s + \ldots \right]$$

$$\epsilon_s = \frac{1}{K_1}$$
Similarly, for other test signals we can find steady state error. \( K_1, K_2, K_3 \) are known as "error coefficients".

**Example 3.11.** Find the dynamic error coefficients of the unity feedback system whose forward transfer function is \( G(s) = \frac{200}{s(s+5)} \). Find the steady state error of the system for the input

**Solution:** Given that

\[
G(s) = \frac{200}{s(s+5)} \\
H(s) = 1 \\
E(s) = \frac{1}{1+G(s)H(s)} \\
E(s) = \frac{s^2 + 5s}{s^2 + 5s + 200} = \frac{5s + s^2}{200 + 5s + s^2} \\
0.025s + 0.004375s^2 - 0.0000218s^3 \\
200 + 5s + s^2 = 5s + s^2 \\
5s = 0.125s^2 = 0.025s^2 \\
0.875s^2 - 0.025 = 0.022s^2 + 0.004375s^3 \\
-0.004375s^4 \\
= 0.0001s^5 + 0.0000218s^6 \\
E(s) = 0.025s + 0.004375s^2 - 0.0000218s^3 \\
or \\
E(s) = 0.025s + 0.004375s^2 + 0.0000218s^3 \\
\text{Take inverse Laplace}
\]

Given that

\[
e(t) = 0.025r(t) + 0.004375r'(t) - 0.0000218 \quad r(t)
\]

\[
r(t) = 4t^2 \\
r(t) = 8t \\
r'(t) = 0 \\
\]

\[
e(t) = 0.2t + 0.035 \\
\]

The steady state error is given by:

\[
\lim_{t \to \infty} e(t) = \lim_{t \to \infty} (0.2t + 0.035)
\]

The dynamic error coefficients are given by

\[
K_1 = \frac{1}{0.025} = 40 \\
K_2 = \frac{1}{0.004375} = 228.57 \\
K_3 = \infty \\
K_4 = 45871.55
\]

**Example 3.12.** Find the dynamic error coefficients of the unity feedback system whose forward path transfer function \( G(s) = \frac{10}{s(s+1)} \).

Find the steady state error to the input \( r(t) = P_e + P_d t + P_2 t^2 \)

**Solution:**

\[
G(s) = \frac{10}{s(s+1)} \\
H(s) = 1
\]

\[
E(s) = \frac{1}{1+G(s)H(s)} = \frac{s + s^2}{10 + s^2 + s^2} = 0.1s + 0.09s^2 - 0.019s^3 \\
E(s) = 0.1s R(s) + 0.09 s^2 R(s) - 0.019 s^2 R(s)
\]

Inverse Laplace

\[
e(t) = 0.1 r(t) + 0.09 \dot{r}(t) - 0.019 \ddot{r}(t) \\
r(t) = P_0 + P_1 t + P_2 t^2 \\
\dot{r}(t) = P_1 + 2P_2 t \\
\ddot{r}(t) = 2P_2 \\
r(t) = 0 \\
e(t) = 0.1 (P_1 + 2P_2 t) + 0.09 \times 2P_2 = 0.1 (P_1 + 2P_2 t) + 0.18 P_2 \\
\]

Steady state error is given by

\[
\lim_{t \to \infty} e(t) = \lim_{t \to \infty} 0.1(P_1 + 2P_2 t) + 0.18 P_2 \\
Dynamic error coeff. are \\
K_1 = 10; \quad K_2 = 11.1; \quad K_3 = -52.63
\]

**SUMMARY**

The steady state error is the difference between input and output of the system during steady state. For accuracy steady state error should be minimum.

\[
e_{SS} = \lim_{s \to 0} sE(s) \\
e_{SS} = \lim_{s \to 0} \frac{R(s)}{1+G(s)H(s)} \\
G(s) H(s) = \frac{K(1+sT_e)}{(1+sT_d)...} \\
G(s) H(s) = \frac{s^n(1+sT_e)}{(1+sT_d)...} \\
m = \text{type of system} \\
m = 0, \text{system is type zero} \\
m = 1, \text{system is type one etc.} \\
K_p = \text{static position error constant} \\
= \lim_{s \to 0} G(s) H(s) \to \text{be used for step input} \\
K_v = \text{Static velocity coefficient} = \lim_{s \to 0} sG(s) H(s) \to \text{be used for ramp input} \\
K_a = \text{static acceleration error constant} \\
= \lim_{s \to 0} s^2G(s) H(s) \to \text{be used for parabolic input}


**Chapter 4**

**Frequency Domain Analysis**

4.1. **INTRODUCTION**

The magnitude and phase relationship between sinusoidal input and steady state output of a system is known as frequency response. It is independent of the amplitude and phase of the input signal. If input signal is

\[ r(t) = X \sin \omega t \]

the output can be written as

\[ c(t) = Y \sin (\omega t + \theta) \]

The main advantage is that the frequency response of a stable open loop system can be obtained experimentally. Apply a sine wave at the input of the system and measure the ratio of the magnitude of output & input as well as the phase difference. Another advantage of this approach is that it is simple and accurate. We can determine experimentally the transfer function of the complicated system. Frequency response tests are not suitable for the systems having large time constant because for each frequency it takes long time to reach the steady state.

\[ G(s) = \frac{C(s)}{R(s)} \quad \text{or} \quad G(j\omega) = \frac{C(j\omega)}{R(j\omega)} \]

\[ \text{or} \quad G(j\omega) = R(s), \quad G(s) \quad \text{or} \quad C(j\omega) = R(j\omega) G(j\omega) \]

The output is the product of input and \|G(s)\|.

For stable system, the sinusoidal output is of the same frequency as input but the amplitude and phase of the output will be different from the input.

\[ G(j\omega) = M \angle \phi \]

where \( M \) is the ratio of amplitudes of output and input sinusoids and is called the gain or magnitude ratio.

\( \phi \) is angle by which output leads the input. \( M \& \phi \) are the function of angular frequency \( \omega \).

4.2. **POLAR PLOT**

The polar plot of a sinusoidal transfer function \( G(j\omega) \) is a plot of the magnitude of \( G(j\omega) \) versus the phase angle of \( G(j\omega) \) on polar coordinates as \( \omega \) is varied from zero to infinity. The polar plot, therefore is the locus of vectors \( G(j\omega) \) as \( \omega \) is varied from zero to infinity. Thus we can express the frequency response function \( G(j\omega) \) in the polar form \( M \angle \phi \) and plot the vector \( M \angle \phi \) in the \( G \)-plane as \( \omega \) varies from zero to infinity. In polar plots the magnitude of \( G(j\omega) \) is plotted as the distance from the origin while the phase angle is measured from positive real axis. Positive phase angle is measured counter clockwise while negative phase angle is measured clockwise from the positive real axis. The polar plot is often called the Nyquist Plot.

The advantage in using a polar plot is that it depicts the frequency response characteristics of a system over the entire frequency range in a single plot. The disadvantage is that the plot does not indicate the contributions of each individual factor of the open loop transfer function.
4.3. PROCEDURE TO SKETCH THE POLAR PLOT

Step 1: Determine the transfer function \( G(s) \) of the system.
Step 2: Put \( S = j\omega \) in the transfer function to obtain \( G(j\omega) \).
Step 3: At \( \omega = 0 \) and \( \omega = \infty \) calculate \( |G(j\omega)| \), by \( \lim_{\omega \to 0} |G(j\omega)| \) and \( \lim_{\omega \to \infty} |G(j\omega)| \).
Step 4: Calculate the phase angle of \( G(j\omega) \) at \( \omega = 0 \) and \( \omega = \infty \) by \( \lim_{\omega \to 0} \angle G(j\omega) \) and \( \lim_{\omega \to \infty} \angle G(j\omega) \).
Step 5: Rationalize the function \( G(j\omega) \) and separate the real and imaginary parts.
Step 6: Equate the imaginary part \( \Im\{G(j\omega)\} \) to zero and determine the frequencies at which the polar plot intersects the real axis and calculate the value \( G(j\omega) \) at the point of intersection by substituting the determined value of frequency in the expression of \( G(j\omega) \).
Step 7: Equate the real part \( \Re\{G(j\omega)\} \) to zero and determine the frequencies at which plots intersect the imaginary axis and calculate the value \( G(j\omega) \) at the point of intersection by substituting the determined value of frequency in the rationalized expression of \( G(j\omega) \).
Step 8: Sketch the polar plot with the help of above information.

1. TYPE 'ZERO' SYSTEM

\[
G(s) = \frac{K}{(1 + ST_1)(1 + ST_2)}
\]

Step 1: Put \( S = j\omega \)

\[
G(j\omega) = \frac{K}{(1 + j\omega T_1)(1 + j\omega T_2)}
\]

Step 2: Taking the limit for the magnitude of \( G(j\omega) \).

\[
\lim_{\omega \to 0} |G(j\omega)| = \frac{K}{\sqrt{1 + (\omega T_1)^2}} \sqrt{1 + (\omega T_2)^2} = K
\]

\[
\lim_{\omega \to \infty} |G(j\omega)| = \frac{K}{\sqrt{1 + (\omega T_1)^2}} \sqrt{1 + (\omega T_2)^2} = 0
\]

Step 3: Taking the limit for the phase angle of \( G(j\omega) \).

\[
\lim_{\omega \to 0} \angle G(j\omega) = \lim_{\omega \to 0} -\tan^{-1} \omega T_1 - \tan^{-1} \omega T_2 = 0
\]

\[
\lim_{\omega \to \infty} \angle G(j\omega) = \lim_{\omega \to \infty} -\tan^{-1} \omega T_1 - \tan^{-1} \omega T_2 = -180^\circ
\]

Step 4: Separating the real and imaginary parts of \( G(j\omega) \).

\[
G(j\omega) = \frac{K}{1 + j\omega T_1(1 + j\omega T_2)}
\]

\[
G(j\omega) = \frac{K(1 - \omega^2 T_1 T_2)}{1 + \omega^2 T_1^2 + \omega^2 T_2^2 + \omega^2 T_1 T_2}
\]

Equating the real part to zero.

\[
\frac{K(1 - \omega^2 T_1 T_2)}{1 + \omega^2 T_1^2 + \omega^2 T_2^2 + \omega^2 T_1 T_2} = 0
\]

\[
\omega^2 = \frac{1}{T_1 T_2} \quad \text{or} \quad \omega = \frac{1}{\sqrt{T_1 T_2}}
\]

The frequency at which plot intersects the imaginary axis is \( \frac{1}{\sqrt{T_1 T_2}} \).

For positive values of frequencies the polar plot intersects the imaginary axis at \( \omega = \infty \) where:

\[
\omega = \frac{1}{\sqrt{T_1 T_2}} \quad \text{and} \quad \omega = \infty
\]

Value of \( G(j\omega) \) when \( \omega = \frac{1}{\sqrt{T_1 T_2}} \)

\[
G(j\omega) = \frac{K\sqrt{T_1 T_2}}{T_1 + T_2} \quad \text{and} \quad \angle G(j\omega) = -90^\circ
\]

\[
\omega = \infty \quad \text{G}(j\omega) = 0 \quad \angle -90^\circ
\]

* Alternative:

Note: Polar plot will intersect \(-90^\circ\) axis so put \( \angle G(j\omega) = -90^\circ \)\;\;\; - \tan^{-1} \omega T_1 - \tan^{-1} \omega T_2 = -90^\circ

\[
\begin{align*}
\omega T_1 + \omega T_2 & = \infty \quad \text{and} \quad \omega T_1 - \omega T_2 = \infty \\
1 - \omega^2 T_1 T_2 & = 0 \quad \text{or} \quad \omega = \frac{1}{\sqrt{T_1 T_2}}
\end{align*}
\]

Put \( \omega = \frac{1}{\sqrt{T_1 T_2}} \)

\[
\frac{K}{(1 + s T_1)(1 + s T_2)} = \frac{K}{(1 + j\omega T_1)(1 + j\omega T_2)}
\]

\[
|G(j\omega)| = \frac{K\sqrt{T_1 T_2}}{T_1 + T_2}
\]
Step 5: Equating the imaginary part to zero

\[ \frac{K \omega (T_1 + T_2)}{1 + \omega^2 T_1^2 + \omega^2 T_2^2 + \omega^4 T_1 T_2} = 0 \]

\[ \omega = 0 \& \omega = \pm \infty \]

When \( \omega = 0 \)

\[ |G(j\omega)| = K \quad G(j\omega) = 0^e \]

When \( \omega = \infty \)

\[ |G(j\omega)| = 0 \quad G(j\omega) = 0^e \]

The coordinates of the intersection points 'A' and 'B'

\[ G(s) = \frac{K}{(1 + ST_1)(1 + ST_2)} \]

\[ G(s) = \frac{K}{\sqrt{T_1 T_2}} \quad \omega = 0 \]

\[ \omega = -90^\circ \]

Fig. 4.1.

2. TYPE ONE SYSTEM

\[ G(s) = \frac{K}{S(1 + ST_1^2)(1 + ST_2^2)} \]

Step 1: Put \( s = j\omega \)

\[ G(j\omega) = \frac{K}{\sqrt{1 + (\omega T_1^2)^2}} \sqrt{1 + (\omega T_2^2)^2} \quad -90^\circ + \tan^{-1} \omega T_1 - \tan^{-1} \omega T_2 \]

Step 2: Taking the limit for the magnitude of \( G(j\omega) \)

\[ \lim_{\omega \to 0} |G(j\omega)| = \lim_{\omega \to 0} \frac{K}{\sqrt{1 + (\omega T_1^2)^2}} \sqrt{1 + (\omega T_2^2)^2} = \infty \]

\[ \lim_{\omega \to \infty} |G(j\omega)| = \lim_{\omega \to \infty} \frac{K}{\sqrt{1 + (\omega T_1^2)^2}} \sqrt{1 + (\omega T_2^2)^2} = 0 \]

Step 3: Taking the limit for the phase angle of \( G(j\omega) \)

\[ \lim_{\omega \to 0} \angle G(j\omega) = \lim_{\omega \to 0} -90^\circ + \tan^{-1} \omega T_1 - \tan^{-1} \omega T_2 = -90^\circ \]

\[ \lim_{\omega \to \infty} \angle G(j\omega) = \lim_{\omega \to \infty} -90^\circ + \tan^{-1} \omega T_1 - \tan^{-1} \omega T_2 = 270^\circ \]

Step 4: Separating the real and imaginary parts

\[ G(j\omega) = \frac{\omega K (T_1 + T_2)}{\omega + \omega^2 (T_1 + T_2^2 + \omega^2 T_1 T_2) + \omega + \omega^2 (T_1 + T_2^2 + \omega^2 T_1 T_2)} \]

\[ \omega = 0 \]

\[ \omega = -90^\circ \]

\[ \omega = \infty \]

The frequency at the point of intersection on real axis is \( \frac{1}{\sqrt{T_1 T_2}} \)

Now calculate the value of \( G(j\omega) \) at this point.

Put \( \omega = \frac{1}{\sqrt{T_1 T_2}} \) in eqn. (A)

\[ G(j\omega) = -K \frac{T_1 T_2}{T_1 + T_2} \quad \angle G(j\omega) = 0^e \]

\[ G(j\omega) = \infty \quad \angle G(j\omega) = 0^e \]

Step 5: Equate the real part to zero

\[ \frac{K \omega^2 T_1 T_2 - K}{\omega + \omega^2 (T_1 + T_2^2 + \omega^2 T_1 T_2)} = 0 \]

\[ \omega = \frac{1}{\sqrt{T_1 T_2}} \quad \& \omega = \pm \infty \]

For positive values of frequencies the polar plot intersects the imaginary axis at \( \omega = \infty \)

\[ G(j\omega) = 0 \quad \angle = 270^\circ \]

Polar plot is shown in fig. 4.2.

From the polar plot it is clear that in type one system the \( j\omega \) term in denominator contributes 90° to the total phase angle. At \( \omega = 0 \), the magnitude is infinity and phase angle - 90°. At \( \omega = \infty \), the magnitude becomes zero & curve converges to origin. At low frequency, the polar plot is asymptotic to a line parallel to negative imaginary axis.
3. TYPE 'TWO' SYSTEM

\[ G(s) = \frac{K}{s^2(1+sT_1)} \]

Put \( s = j\omega \)

\[ G(j\omega) = \frac{K}{(j\omega)^2(1+j\omega T_1)(1+j\omega T_2)} \]

\[ G(j\omega) = \frac{K}{\omega^2 \sqrt{1 + (\omega T_1)^2}} < -180^\circ - \tan^{-1} \omega T_1 \]

\[ \lim_{\omega \to 0} |G(j\omega)| = \frac{K}{\omega^2} \sqrt{1 + (\omega T_1)^2} < \infty \]

\[ \lim_{\omega \to 0} \angle G(j\omega) = -180^\circ - \tan^{-1} \omega T_1 \]

\[ \lim_{\omega \to \infty} |G(j\omega)| = \frac{K}{\omega^2} \sqrt{1 + (\omega T_2)^2} = 0 \]

\[ \lim_{\omega \to \infty} \angle G(j\omega) = -180^\circ - \tan^{-1} \omega T_2 \]

The presence of \( s^2 \) in the denominator contributes a constant -180° to the angle of \( G(j\omega) \) for all frequencies.

The polar plot is a smooth curve whose angle decreases continuously from -180° to -270°. The plot is shown in Fig. 4.3.

From the polar plot it is clear that at \( \omega = 0 \), magnitude is infinity & phase angle -180°, at \( \omega = \infty \) magnitude is zero & at low frequencies the polar is asymptotic to a line parallel to negative real axis.

INTRODUCTION OF ADDITIONAL POLE

\[ G(s) = \frac{K}{s^2(1+sT_1)(1+sT_2)} \]

Put \( s = j\omega \)

\[ G(j\omega) = \frac{K}{(j\omega)^2(1+j\omega T_1)(1+j\omega T_2)} \]

4.4. POLAR PLOTS OF SOME STANDARD TYPE FUNCTION

1. \( G(s) = \frac{1}{s} \)

Put \( s = j\omega \)

\[ G(j\omega) = \frac{1}{j\omega} \]

\[ G(j\omega) = \frac{1}{\omega} < -\tan^{-1} \infty \]

\[ \lim_{\omega \to 0} |G(j\omega)| = \frac{1}{\omega} \]

\[ \lim_{\omega \to \infty} |G(j\omega)| = \infty \]

\[ \lim_{\omega \to 0} \angle G(j\omega) = -90^\circ \]

\[ \lim_{\omega \to \infty} \angle G(j\omega) = 0 \]

Fig. 4.4.

-270°

-180°

-90°

\( \omega \) increasing

Re

Fig. 4.3.

180°

90°

\( \omega \) increasing

Re

Fig. 4.5.
Example 4.1. Sketch the polar plot for $G(s) = \frac{1}{s(s+1)}$

Solution:

$$G(s) = \frac{1}{s(s+1)}$$

Put $s = j\omega$

$$G(j\omega) = \frac{1}{j\omega (1 + j\omega)}$$

$$G(j\omega) = \frac{1}{\omega \sqrt{1 + \omega^2}} \angle -90^\circ - \tan^{-1}\omega$$

Apply limit to the magnitude of $G(j\omega)$

$$\lim_{\omega \to 0} |G(j\omega)| = \lim_{\omega \to 0} \frac{1}{\omega \sqrt{1 + \omega^2}} = \infty$$

$$\lim_{\omega \to \infty} |G(j\omega)| = \lim_{\omega \to \infty} \frac{1}{\omega \sqrt{\omega^2 + 1}} = 0$$

Taking limit for the magnitude of $G(j\omega)$

$$\lim_{\omega \to 0} G(j\omega) = \lim_{\omega \to 0} \frac{1}{\omega \sqrt{1 + \omega^2}} = \infty$$

$$\lim_{\omega \to \infty} G(j\omega) = \lim_{\omega \to \infty} \frac{1}{\omega \sqrt{\omega^2 + 1}} = 0$$

Taking limit for the phase angle of $G(j\omega)$

$$\lim_{\omega \to 0} G(j\omega) = \lim_{\omega \to 0} \frac{1}{\omega \sqrt{1 + \omega^2}} = -90^\circ - \tan^{-1}\omega$$

$$\lim_{\omega \to \infty} G(j\omega) = \lim_{\omega \to \infty} \frac{1}{\omega \sqrt{\omega^2 + 1}} = -90^\circ = -180^\circ$$

Separate the real and imaginary part of $G(j\omega)$

$$G(j\omega) = \frac{10}{j\omega (j\omega + 1)} = -\omega^2 + j\omega$$

$$= \frac{10}{-\omega^2 - j\omega + 10} \left( \frac{-\omega^2 - j\omega}{\omega^2 - \omega^2} \right)$$

$$= \frac{10(-\omega^2 - j\omega)}{\omega^2 + \omega^2}$$

Example 4.2. Sketch the polar plot of $G(s) = \frac{10}{s(s+1)}$

Solution:

$$G(s) = \frac{10}{s(s+1)}$$

Put $s = j\omega$

$$G(j\omega) = \frac{10}{j\omega (j\omega + 1)}$$

$$G(j\omega) = \frac{10}{\omega \sqrt{1 + \omega^2}} \angle -90^\circ - \tan^{-1}\omega$$

Taking limit for the magnitude of $G(j\omega)$

$$\lim_{\omega \to 0} |G(j\omega)| = \lim_{\omega \to 0} \frac{10}{\omega \sqrt{\omega^2 + 1}} = \infty$$

$$\lim_{\omega \to \infty} |G(j\omega)| = \lim_{\omega \to \infty} \frac{10}{\omega \sqrt{\omega^2 + 1}} = 0$$

Taking limit for the phase angle of $G(j\omega)$

$$\lim_{\omega \to 0} G(j\omega) = \lim_{\omega \to 0} \frac{10}{\omega \sqrt{\omega^2 + 1}} = -90^\circ - \tan^{-1}\omega$$

$$\lim_{\omega \to \infty} G(j\omega) = \lim_{\omega \to \infty} \frac{10}{\omega \sqrt{\omega^2 + 1}} = -90^\circ = -180^\circ$$

Separate the real and imaginary part of $G(j\omega)$

$$G(j\omega) = \frac{10}{j\omega (j\omega + 1)} = -\omega^2 + j\omega$$

$$= \frac{10(-\omega^2 - j\omega)}{\omega^2 + \omega^2}$$
Equating the imaginary part to zero

\[
\frac{10\omega}{\omega^2 + \omega^2} = 0 \quad \therefore \omega = \infty
\]

Put \(\omega = \infty\) in eqn (A) \(G(j\omega) = 0\), i.e. plot intersects the real axis at the origin.

Equating the real part to zero

\[
\frac{-10\omega^2}{\omega^2 + \omega^2} = 0
\]

or,

\[
\frac{-10}{\omega^2 + 1} = 0
\]

\(\omega = \infty\)

Put \(\omega = \infty\) in equation (A)

\(G(j\omega) = 0\) i.e. the plot intersects the imaginary axis at the origin. The required polar plot is shown in Fig. 4.9.

Example 4.3. Sketch the polar plot for \(G(s) = \frac{20}{s(s + 1)(s + 2)}\).

Solution:

\[
G(s) = \frac{20}{s(s + 1)(s + 2)}
\]

Put \(s = j\omega\)

\[
G(j\omega) = \frac{20}{j\omega(1 + j\omega)(j\omega + 2)}
\]

Equating the imaginary part to zero

\[
\frac{20(\omega^2 - 2\omega)}{(\omega^4 + \omega^2)(4 + \omega^2)} = 0
\]

For the positive values of frequencies polar plot intersects the real axis at \(\omega = \sqrt{2}\) and \(\omega = \infty\) for which \(G(j\omega) = \frac{-10}{\omega^2} \leq 0^\circ\) and \(0 \leq \theta^\circ\)

Equating the real part to zero

\[
\frac{-60\omega^2}{(\omega^4 + \omega^2)(4 + \omega^2)} = 0
\]

\(\omega = \infty\)

The polar plot intersects the imaginary axis at \(\omega = \infty\) for which \(G(j\omega) = 0 \leq -270^\circ\).

The required polar plot is shown in Fig. 4.10.

Example 4.4. Sketch the polar plot for \(G(s) = \frac{s}{1 + sT}\).

Solution:

\[
G(s) = \frac{j\omega}{1 + j\omega T}
\]

\[
G(j\omega) = \frac{\omega}{\sqrt{1 + \omega^2 T^2}} \leq 90^\circ - \tan^{-1} \omega T
\]

Taking the limit for the magnitude of \(G(j\omega)\)

\[
\lim_{\omega \to \infty} |G(j\omega)| = \lim_{\omega \to 0} \frac{\omega}{\sqrt{1 + \omega^2 T^2}} = \infty
\]

\[
\lim_{\omega \to 0} |G(j\omega)| = \lim_{\omega \to \infty} \frac{20(\omega^2 - 2\omega)}{(\omega^4 + \omega^2)(4 + \omega^2)} = -270^\circ
\]

Separating the real and imaginary part of \(G(j\omega)\)

\[
G(j\omega) = \frac{-60\omega^2}{(\omega^4 + \omega^2)(4 + \omega^2)} + j\frac{20(\omega^2 - 2\omega)}{(\omega^4 + \omega^2)(4 + \omega^2)}
\]

Taking the limit for the magnitude of \(G(j\omega)\)

\[
\lim_{\omega \to \infty} |G(j\omega)| = \lim_{\omega \to 0} \frac{\omega}{\sqrt{1 + \omega^2 T^2}} = 0
\]

Taking the limit for the phase angle of \(G(j\omega)\)

\[
\lim_{\omega \to \infty} \angle G(j\omega) = \lim_{\omega \to 0} \frac{\omega}{\sqrt{1 + \omega^2 T^2}} = \frac{\pi}{2}
\]
\[ \lim_{\omega \to \pm \infty} G(j\omega) = \lim_{\omega \to \pm \infty} \frac{90^\circ - \tan^{-1} \omega T}{\omega T} = 0^\circ \]

Separating the real and imaginary terms of \( G(j\omega) \):

\[
G(j\omega) = \frac{j\omega T}{1 + j\omega T} \quad \Rightarrow \quad \frac{\omega^2 T^2}{1 + \omega^2 T^2} + \frac{j\omega}{1 + \omega^2 T^2}
\]

Equating the imaginary part to zero:

\[
\frac{\omega^2 T^2}{1 + \omega^2 T^2} = 0 \quad \Rightarrow \quad \omega = \infty
\]

Polar plot intersects the real axis at \( \omega = \infty \) at which \( G(j\omega) = \frac{1}{T} \angle 0^\circ \)

Equate the real part to zero:

\[
\frac{\omega^2 T^2}{1 + \omega^2 T^2} = 0 \quad \Rightarrow \quad \omega = 0
\]

Polar plot intersects the imaginary axis at \( \omega = 0 \) at which \( G(j\omega) = 0 \angle 90^\circ \)

The required polar plot is given below.

\[
G(s) = \frac{s}{s + 1}
\]

**Fig. 4.11.**

4.5 ALTERNATIVE METHOD

The other method to draw the polar plot is the tip of the vector \( M(\omega) \) \( e^{j\phi(\omega)} \) in the G-plane as \( \omega \) varies from zero to infinity.

The first step is to calculate the magnitude ratio \( M \) and phase shift \( \phi \) for different values of \( \omega \).

Consider the following examples.

**Example 4.5. Sketch the polar plot for**

\[
G(s) = \frac{10(s + 1)}{s + 10}
\]

Solution: Put \( s = j\omega \)

\[
|G(j\omega)| = \frac{10\sqrt{1 + \omega^2}}{\sqrt{\omega^2 + 100}}
\]

\[
\angle G(j\omega) = \tan^{-1} \omega - \tan^{-1} \frac{\omega}{10}
\]

**Table 4.1.**

<table>
<thead>
<tr>
<th>( \omega )</th>
<th>( M )</th>
<th>( \phi )</th>
<th>( M \cos \phi + jM \sin \phi )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>1</td>
<td>0^\circ</td>
<td>1 + j0</td>
</tr>
<tr>
<td>1</td>
<td>1.41</td>
<td>39.29^\circ</td>
<td>1.09 + j0.49</td>
</tr>
<tr>
<td>3.16</td>
<td>3.16</td>
<td>54.90^\circ</td>
<td>1.81 + j2.58</td>
</tr>
<tr>
<td>4</td>
<td>3.85</td>
<td>54.29^\circ</td>
<td>2.24 + j3.106</td>
</tr>
<tr>
<td>7</td>
<td>5.8</td>
<td>46.86^\circ</td>
<td>3.96 + j4.2</td>
</tr>
<tr>
<td>10</td>
<td>7.1</td>
<td>39.30^\circ</td>
<td>5.5 + j4.5</td>
</tr>
<tr>
<td>20</td>
<td>8.96</td>
<td>23.70^\circ</td>
<td>8.2 + j3.6</td>
</tr>
<tr>
<td>50</td>
<td>9.81</td>
<td>10.16^\circ</td>
<td>9.67 + j1.73</td>
</tr>
<tr>
<td>100</td>
<td>9.95</td>
<td>5.14^\circ</td>
<td>9.9 + j0.89</td>
</tr>
<tr>
<td>200</td>
<td>9.99</td>
<td>2.55^\circ</td>
<td>9.97 + j0.45</td>
</tr>
</tbody>
</table>

The corresponding polar plot is shown on the graph paper in fig. 4.12.

**Example 4.6. Sketch the polar plot of**

\[
G(s) = \frac{20s}{(s + 1)(s + 10)}
\]

Solution: Put \( s = j\omega \)

\[
G(j\omega) = \frac{20(j\omega)}{(j\omega + 1)(j\omega + 10)}
\]

**Table 4.2.**

<table>
<thead>
<tr>
<th>( \omega )</th>
<th>( M )</th>
<th>( \phi )</th>
<th>( M \cos \phi + jM \sin \phi )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.2</td>
<td>0.4</td>
<td>77.56^\circ</td>
<td>0.08 + j0.4</td>
</tr>
<tr>
<td>0.5</td>
<td>0.9</td>
<td>60.58</td>
<td>0.44 + j0.78</td>
</tr>
<tr>
<td>0.7</td>
<td>1.147</td>
<td>51.0</td>
<td>0.72 + j0.89</td>
</tr>
<tr>
<td>1.0</td>
<td>1.41</td>
<td>39.28</td>
<td>1.09 + j0.9</td>
</tr>
<tr>
<td>2.0</td>
<td>1.79</td>
<td>15.3</td>
<td>1.72 + j0.47</td>
</tr>
<tr>
<td>3.0</td>
<td>1.82</td>
<td>1.74</td>
<td>1.8 + j0.05</td>
</tr>
<tr>
<td>4.0</td>
<td>1.8</td>
<td>-7.8^\circ</td>
<td>1.78 - j0.24</td>
</tr>
<tr>
<td>5.0</td>
<td>1.78</td>
<td>-15.26^\circ</td>
<td>1.71 - j0.46</td>
</tr>
<tr>
<td>10.0</td>
<td>1.41</td>
<td>-39.3</td>
<td>1.09 - j0.9</td>
</tr>
<tr>
<td>20.0</td>
<td>0.89</td>
<td>-60.5</td>
<td>0.43 - j0.77</td>
</tr>
<tr>
<td>30.0</td>
<td>0.63</td>
<td>-69.5</td>
<td>0.22 - j0.6</td>
</tr>
<tr>
<td>40.0</td>
<td>0.48</td>
<td>-74.56</td>
<td>0.12 - j0.5</td>
</tr>
<tr>
<td>50.0</td>
<td>0.39</td>
<td>-77.5</td>
<td>0.08 - j0.4</td>
</tr>
<tr>
<td>60.0</td>
<td>0.32</td>
<td>-79.5</td>
<td>0.05 - j0.3</td>
</tr>
</tbody>
</table>

Corresponding polar plot is shown in fig. 4.13.
4.6. PHASE MARGIN, GAIN MARGIN AND STABILITY ON POLAR PLOT

The stability of control system is measured from the measurement of quantities like 'phase margin' and 'gain margin'. These quantities are useful to design the compensating network for unstable systems, i.e., for the improvement of stability of an existing control system.

In above fig. polar plot intersects the negative real axis at a distance of 'x' from the origin. The frequency where it crosses the negative real axis is called "phase cross-over frequency." The point where the polar plot intersects the negative real axis is called "phase crossover point."

Phase Margin

The phase margin is that amount of additional phase lag at the gain crossover frequency required to bring the system to the verge of instability.

A circle with radius equal to unity is drawn and it intersects the polar plot at point 'A' where the frequency is \( \omega_p \). This frequency is called "gain cross over frequency." In other words the gain crossover frequency is the frequency at which \( |G(j\omega_p)| \), the magnitude of the open-loop transfer function is unity.

The phase margin is equal to \( 180^\circ \) plus the angle of \( G(j\omega) \) at the gain crossover point i.e.

\[
\phi_m = 180^\circ + \phi
\]

\[
\left( r - \phi + 180 = 360, \quad r = 360 + \phi - 180 = 180 + \phi \right)
\]

The angle from the negative real axis to the line OA is the phase margin. The phase margin is positive for \( \phi_m > 0 \) and negative \( \phi_m < 0 \). For minimum phase system to be stable, the phase margin must be positive. For unstable system the gain crossover point would not be found in the third quadrant of \( G(\omega) \) plane. When phase crossover point is to the left of \((-1 + j0)\) the phase margin is negative and the closed loop system is unstable.

Gain Margin

The gain margin is the reciprocal of the magnitude \( |G(j\omega)| \) at the frequency at which the phase angle is \(-180^\circ\).

\[
\text{Gain margin (GM)} = \frac{1}{|G(j\omega)|}
\]

\( ^* \) minimum phase system is describe in Art 4.9.
In terms of decibels
\[ K_p \text{ dB} = 20 \log K_p = -20 \log |G(j\omega)| \]
(i) If plot does not intersect the negative real axis \( |G(j\omega)| = 0 \)
\[ K_p = \infty \]
(ii) If the plot intersects the negative real axis between the origin and \((-1 + j0)\)
\[ O < |G(j\omega)| < 1 \]
\[ K_p > 0 \text{ db} \]
(iii) If the plot intersects the negative real axis at \((-1 + j0)\)
\[ |G(j\omega)| = 1 \]
\[ K_p = 0 \text{ db} \]
(iv) If the plot encloses the point \((-1 + j0)\) \( |G(j\omega)| > 1 \)
\[ K_p < 0 \text{ db} \]
When the plot passes through the point \((-1 + j0)\), the gain margin is zero it means that the loop gain can no longer be increased because the system is at the verge of instability.
If the gain margin is positive it means that the system is stable and negative gain margin means the system is unstable.

\[ \frac{1}{K_p} \]

\[ \phi = -180^\circ \]
\[ -180^\circ = -90^\circ - \tan^{-1} \omega - \tan^{-1} 0.2 \omega \]
\[ 90^\circ = \tan^{-1} \omega + \tan^{-1} 0.2 \omega \]
\[ \tan 90^\circ = \tan (\tan^{-1} \omega + \tan^{-1} 0.2 \omega) \]
\[ \infty = 1 - 0.2 \omega^2 \]
or, \[ 1 - 0.2 \omega^2 = 0 \]
or, \[ \omega = 2.236 \text{ rad/sec.} \]
This is the phase crossover frequency. Now find the magnitude of \( G(j\omega) H(j\omega) \) at this frequency.
\[ |G(j\omega) H(j\omega)| = x = \frac{K}{j2.236((1+j2.236)^2 + 5j2.236)} \]
\[ x = \frac{K}{30} \]
\[ \therefore \text{if } K = 30, \text{ } x = 1 \text{ and system will be marginally stable} \]
\[ K > 30, \text{ } x > 1 \text{ and system will be unstable} \]
\[ K < 30, \text{ } x < 1 \text{ and system will be stable} \]

4.7. INVERSE POLAR PLOT

The inverse polar plot of \( G(s) \) is a graph of \( \frac{1}{G(j\omega)} \) as a function of \( \omega \).

For example, if \( G(j\omega) = \frac{1}{j\omega} \) then \( G(j\omega)^{-1} = j\omega \)
\[ G(j\omega)^{-1} = \omega + 90^\circ \]
\[ \lim \frac{G(j\omega)^{-1}}{\omega} = 0 \]
\[ \lim \frac{G(j\omega)^{-1}}{\omega} = \infty \]

The inverse polar plot is shown in the fig 4.17.

Example 4.7. Sketch the inverse polar plot of \( G(j\omega) = j\omega T^2 \) as \( \omega \) increases

Solution:
\[ G(j\omega)^{-1} = \frac{1}{G(j\omega)} = \frac{1 + j\omega T}{j\omega T} = \frac{1}{j\omega} + 1 \]
$$\lim_{s \to 0} [G(j\omega)^{-1}] = \infty$$

$$\lim_{s \to \infty} [G(j\omega)^{-1}] = 1$$

$$\lim_{s \to 0} \frac{G(j\omega)}{1 + 20N} = 0$$

$$\lim_{s \to \infty} \frac{G(j\omega)}{1 + 20N} = 0$$

The polar plot is shown in Fig. 4.18.

**Example 4.8.** Sketch the inverse polar plot of

$$\frac{\text{Output}}{\text{Input}} = \frac{1 + ST}{ST}$$

**Solution:**

$$G(s)^{-1} = \frac{1}{G(s)} = \frac{sT}{1 + sT}$$

Put

$$s = j\omega$$

$$G(j\omega)^{-1} = \frac{1}{G(j\omega)} = \frac{j\omega T}{1 + j\omega T}$$

$$\lim_{\omega \to 0} \frac{G(j\omega)^{-1}}{1 + 20N} = 0$$

$$\lim_{\omega \to \infty} \frac{G(j\omega)^{-1}}{1 + 20N} = 0$$

The inverse polar plot is shown in Fig. 4.19.

**4.8 BODE PLOT**

Bode plot is a graphical representation of the transfer function for determining the stability of the control system. Bode plot consists of two separate plots. One is a plot of the logarithm of the magnitude of a sinusoidal transfer function, the other is a plot of the phase angle, both plots are plotted against the frequency. The curves are drawn on semilog graph paper, using the log scale for frequency and linear scale for magnitude (in decibels) or phase angle (in degrees). The magnitude is represented in dB.

(i) 20 log_{10} |G(j\omega)| V, log\omega.
(ii) Phase shift V, logomega.

The main advantage of using Bode plot is that multiplication of magnitudes can be converted into addition.

Consider open loop transfer function of a closed loop control system

$$G(s) H(s) = \frac{K(1 + sT_1)(1 + sT_2)}{s^2(1 + sT_1)(1 + sT_2)}$$

Put

$$s = j\omega$$

$$G(j\omega) H(j\omega) = \frac{K(1 + j\omega T_1)(1 + j\omega T_2)}{(j\omega)^N(1 + j\omega T_1)(1 + j\omega T_2)}$$

$$20 \log_{10} |G(j\omega)| = 20 \log_{10} \left( 1 + \frac{1}{(j\omega)^N} \right)$$

$$= 20 \log_{10} (j\omega)^{-N}$$

$$= -20 N \log_{10} (\omega)$$

where

$$N = 1, 2, 3, \ldots$$

The plot M vs log_{10} \omega is a straight line. For N = 1 the line has a slope of -20 dB/decade and angle -90°. For N = 2, the slope of the line will be -40 dB/decade and angle will be -180° and so on.
Put the different values of \( \omega \), we will get \( |G(j\omega)| \) considering the following two cases.

(i) For \( \omega T \ll 1 \) (very low frequencies)

\[
-20 \log_{10} \sqrt{1 + \omega^2 T^2} = -20 \log_{10} \sqrt{1} = 0
\]

\[ M = 0 \text{ for } \omega T \ll 1 \text{ or } \omega \leq \frac{1}{T} \]

(ii) For \( \omega T \gg 1 \) (very high frequencies)

\[
-20 \log_{10} \sqrt{1 + \omega^2 T^2} = -20 \log_{10} \omega T \text{ for } \omega \gg 1/T
\]

Hence, \( M \) vs \( \log_{10} \omega \) has two parts

(i) One part having \( M = 0 \) for \( \omega \ll 1/T \)

(ii) In other part \( M \) varies as a straight line with slope of -20 db/decade for

\[ \omega \gg \frac{1}{T} \]

\[ \omega = \frac{1}{T} \] is called break frequency or corner frequency

\[ M = -20 \log_{10} \omega T = -20 (\log_{10} \omega + \log_{10} T) \]

\[ M = -20 \log_{10} \omega - 20 \log_{10} T \]

\[ = -20 \log_{10} 0 + 20 \log_{10} 1/T \] \hspace{1cm} (4.3)

The above two parts of the graph intersect 0 db axis is determined by equating the eq(4.3) to zero

\[ 0 = -20 \log_{10} \omega + 20 \log_{10} 1/T \]

\[ \omega = 1/T \] is called break frequency.

\[ \omega = 1/T \] is called break frequency.

\[ \omega \gg 1/T \] is called break frequency.
(ii) When \( aT \gg 1 \)

\[
M = 20 \log_{10} aT \\
M = 20 \log_{10} a = 20 \log_{10} \frac{a}{T} \\
= 20 \log_{10} a - 20 \log_{10} \frac{1}{T}
\]

equate the above equation to zero

\[
0 = 20 \log_{10} a - 20 \log_{10} \frac{1}{T}
\]

\[
a \approx \frac{1}{T} \text{ corner frequency.}
\]

Thus, the two parts of the graph intersect the '0 dB' axis at \( a \approx \frac{1}{T} \). The second part is a straight line having the slope of +20 dB/decade.

**Phase Angle Plot**

\[
\phi = \frac{\angle G(j\omega)}{20} \approx \tan^{-1} \frac{\omega}{aT}
\]

(i) At very low frequencies \( aT \) is very very small

\[
\phi = \tan^{-1} (0) = 0
\]

(ii) at \( aT = 1 \)

\[
\phi = \tan^{-1} (1) = 45^\circ
\]

(iii) at very high frequencies

\[
\phi = \tan^{-1} (\infty) = 90^\circ
\]

Thus, the value of \( \phi \) gradually changes from 0° to 90° as \( \omega \) increases from 0 to very high values.

\[\text{Fig. 4.24.}\]

Case 6: General second order system

\[
G(s) = \frac{s^2 + 2\zeta\omega_n s + \omega_n^2}{s^2 + 2\xi\omega_n s + \omega_n^2}
\]

Put \( s = j\omega \)

\[
G(j\omega) = \frac{\omega_n^2}{(j\omega)^2 + 2\xi\omega_j (j\omega) + \omega_n^2} = \frac{\omega_n^2}{-\omega^2 + j2\xi\omega_n \omega + \omega_n^2}
\]

\[
20 \log_{10} |G(j\omega)| = 20 \log_{10} \frac{1}{|(\omega/\omega_n)^2 + j2\xi\omega_n \omega/\omega_n|}
\]

\[
= -20 \log_{10} \left[ \left( \frac{\omega}{\omega_n} \right)^2 + j2\xi\omega_n \omega/\omega_n \right]^2
\]

Suppose \( \frac{\omega}{\omega_n} = u \)

\[
20 \log_{10} |G(j\omega)| = M = -20 \log_{10} \sqrt{1 + (u^2) - 4\xi^2 u^2}
\]

Consider the two cases

1. \( u << 1 \) \( \Rightarrow \frac{\omega}{\omega_n} << 1 \)

\[
M = -20 \log_{10} \sqrt{1} = 0 \text{ db.}
\]

2. \( u >> 1 \) \( \Rightarrow \frac{\omega}{\omega_n} >> 1 \)

\[
M = -20 \log_{10} \sqrt{(u^2 - 1)} = -20 \log_{10} u^2 = -40 \log_{10} u
\]

So, it is a straight line having slope of -40 dB/dec. and passing through the point \( u \).

Therefore, the asymptotic plot consists of

(i) \( M = 0 \) \( u << 1 \)

(ii) \( M = -40 \log_{10} u \) \( u >> 1 \)

**Phase Angle Plot**

\[
\phi = \frac{\angle G(j\omega)}{20} = -\tan^{-1} \frac{2\xi u}{1 - u^2}
\]

(i) for small value of \( u \), \( u^2 \) is small

\[
\phi = -\tan^{-1} 2\xi u
\]

(ii) for large value of \( u \), \( u^2 >> 1 \)

\[
\phi = +\tan^{-1} \frac{2\xi}{u}
\]

(iii) when \( u = 1 \)

\[
\phi = -\tan^{-1} \infty = -90^\circ
\]
Initial Slope of Bode Plot

Let 
\[ G(s)H(s) = \frac{K}{s^N} \]

Put 
\[ s = j\omega \]

\[ G(j\omega)H(j\omega) = \frac{K}{(j\omega)^N} \]

\[ 20 \log_{10} |G(j\omega)H(j\omega)| = 20 \log_{10} \left| \frac{K}{(j\omega)^N} \right| \]

\[ = 20 \log_{10} K - 20 N \log_{10} \omega \quad \text{(4.4)} \]

1. For \( N = 0 \) (Type zero system)

\[ 20 \log_{10} |G(j\omega)H(j\omega)| = 20 \log_{10} K \]

This is a straight line. The graph is shown in fig. 4.26.

2. For \( N = 1 \) (type one system)

Put \( N = 1 \) in equation (4.4)

\[ 20 \log_{10} |G(j\omega)H(j\omega)| = 20 \log_{10} K - 20 \log_{10} \omega \]

Intersection with 0 db axis

\[ 0 = 20 \log_{10} K - 20 \log_{10} \omega \]

\[ K = \omega \]

locate \( \omega = K \) on 0 db axis and at this point draw a line of -20 db/decade produce it till it intersect the y-axis that will be the starting point on Bode plot.

3. For \( N = 2 \) (type two system)

Put \( N = 2 \) in eq (4.4)

\[ 20 \log_{10} |G(j\omega)H(j\omega)| = 20 \log_{10} K - 20.2 \log_{10} \omega \]

\[ = 20 \log_{10} K - 40 \log_{10} \omega \]

4.9. MINIMUM PHASE SYSTEMS AND NON-MINIMUM PHASE SYSTEMS

The transfer functions having no poles and zeros in the right half s-plane are called minimum phase transfer functions. Systems with minimum phase transfer functions are called minimum phase systems.

The transfer functions having poles and/or zeros in the right half s-plane are called non-minimum phase transfer functions. Systems with non-minimum phase transfer functions are called non-minimum-phase systems.

Let

\[ G_1(j\omega) = \frac{1 + j\omega T_1}{1 + j\omega T_2} \]

\[ G_2(j\omega) = \frac{1 - j\omega T_1}{1 + j\omega T_2} \]

The transfer function given by eq (4.5) is a minimum phase transfer function and transfer function given by eq (4.6) is non-minimum phase type transfer function.
For minimum phase systems, the magnitude and phase angle plots are uniquely related. It is known that if the magnitude curve is specified for the frequency from zero to infinity, then the phase curve is uniquely related. This is not applicable for non-minimum phase systems.

For minimum phase systems, the phase angle at \( \omega = \infty \) is \(-90^\circ (q-p)\), where \( q \) and \( p \) are the degree of numerator and denominator polynomials of transfer function. For both minimum and non-minimum phase systems, the slope is \(-20 (q-p) \text{ dB/decade}\). But the phase angle for non-minimum phase systems is different from \(-90^\circ (q-p)\). Thus it is possible to determine whether the system is minimum or non-minimum phase system. If the slope of the log-magnitude curve at infinity is \(-20 (q-p) \text{ dB/dec} \) and the phase angle is equal to \(-90^\circ (q-p)\) at \( \omega = \infty \), then the system is minimum phase otherwise not.

Non-minimum phase systems are slow in response. In control systems, excessive phase lag has to be avoided.

### 4.10. PROCEDURE FOR DRAWING THE BODE PLOTS

Consider the transfer function

\[
G(s) = \frac{K(1+Ts_p)}{(1+sT_s)(1+Ts_2)}
\]

Where \( N \) is the number of poles at the origin, i.e., \( N \) defines the type of system.

- For type zero system \( K = K_p \)
- For type one system \( K = K_p \)
- For type two system \( K = K_p \)

In above transfer function put \( s = j\omega \)

\[
G(j\omega) = \frac{K(1+j\omega T_p)}{(1+j\omega T_s)(1+j\omega T_2)}
\]

20\log_{10} |G(j\omega)| = 20\log K + 20\log \sqrt{1+\omega^2 T_s^2} + 20\log \sqrt{1+\omega^2 T_p^2} - 20N \log \omega

Phase angle

\[
\angle G(j\omega) = \tan^{-1} \frac{\omega T_p}{1-\omega^2 T_1^2} + \tan^{-1} \frac{\omega T_2}{2\omega T_2} + \ldots N(90^\circ) - \tan^{-1} \frac{\omega T_1}{1-\omega^2 T_2^2} - \tan^{-1} \frac{2\omega T_1 T_2}{\omega^2 - T_2^2}
\]

**Step 1:** Identify the corner frequency.

**Step 2:** Draw the asymptotic magnitude plot. The slope will change at each corner frequency by \(-20 \text{ dB/dec} \) for zero and \(-40 \text{ dB/dec} \) for pole. For complex conjugate pole and zero the slope will change by \( \pm 40 \text{ dB/dec} \).

**Step 3:**

(i) For type zero system, draw a line up to first (lowest) corner frequency having 0 dB dec slope.

(ii) For type one system, draw a line having slope \(-20 \text{ dB/dec} \) up to \( \omega = K \). Mark first (lowest) corner frequency.

(iii) For type two system, draw the line having slope \(-40 \text{ dB/dec} \) up to \( \omega = \sqrt{K} \) and so on.

**Step 4:** Draw a line up to second corner frequency by adding the slope of next pole or zero to the previous slope and corner frequency.

**Step 5:** Calculate phase angle for different values of \( \omega \) from the equation 4.10 and join all points.

### 4.11. PHASE MARGIN & GAIN MARGIN

![Gain crossover frequency](image1)

**Gain crossover frequency**

\[
\text{Gain crossover frequency} = \frac{1}{G(j\omega) H(j\omega)_{\omega = \omega_c}}
\]

where \( \omega_c \) is phase cross-over frequency.

Generally, G.M. is expressed in decibels.
In decibels  
\[ \text{G.M.} = 20 \log_{10} \left( \left| \frac{G(j\omega)H(j\omega)}{|G(j\omega)H(j\omega)|}_{\omega = \omega_0} \right| \right) \]

or,
\[ \text{G.M.} = -20 \log_{10} |G(j\omega)H(j\omega)|_{\omega = \omega_0} \]

**Phase Margin:** For gain the additional phase lag can be introduced without affecting the magnitude plot. Therefore, phase margin can be defined as the amount of additional phase lag which can be introduced in the system till system reaches on the verge of instability called as phase margin (P.M.). Mathematically phase margin can be defined as
\[ \text{P.M.} = \left[ \frac{\angle G(j\omega)H(j\omega)}{\omega = \omega_0} \right] - (-180^\circ) \]
\[ \text{P.M.} = 180^\circ + \frac{\angle G(j\omega)H(j\omega)}{\omega = \omega_0} \]

where \( \omega_0 \) = Gain cross-over frequency.

The above expressions for gain margin and phase margin can be directly used for mathematical determination of G.M. & P.M. For example, for the given transfer function \( G(s)H(s) = \frac{2}{s(1+0.5s)(1+0.05s)} \) determine phase cross-over frequency, gain margin, gain cross-over frequency and phase margin.

According to definition, the phase cross-over frequency is the frequency at which phase angle of \( G(j\omega)H(j\omega) \) is \(-180^\circ\).

For given transfer function
\[ \frac{G(j\omega)H(j\omega)}{\omega = \omega_0} = -90^\circ - \tan^{-1} 0.5 \omega - \tan^{-1} 0.05 \omega \]

According to definition,
\[ \text{P.M.} = 180^\circ + \frac{\angle G(j\omega)H(j\omega)}{\omega = \omega_0} \]

or, \[ \tan^{-1} 0.5 \omega + \tan^{-1} 0.05 \omega = 90^\circ \]

taking tangent on both sides
\[ \tan^{-1} 0.5 \omega + \tan^{-1} 0.05 \omega = \tan 90^\circ \]
\[ 0.5 \omega = 0.05 \omega \]
\[ 1 = 0.5 \omega 0.05 \omega \]
\[ 1 = 0.025 \omega \]
\[ \omega = 6.32 \text{ rad/sec.} \]

This is the phase cross-over frequency.
\[ \omega_0 = 6.32 \text{ rad/sec.} \]

Now calculate magnitude at this frequency
\[ |G(j\omega)H(j\omega)|_{\omega = \omega_0} = 0.0918 \]

The gain cross-over frequency the magnitude of \( G(j\omega)H(j\omega) \) is unity.

\[ |G(j\omega)H(j\omega)| = \frac{2}{\omega \sqrt{1 + 0.25 \omega} \sqrt{1 + 0.0025 \omega}} \]

or
\[ \frac{1}{\omega \sqrt{1 + 0.25 \omega} \sqrt{1 + 0.0025 \omega}} \]

\[ \phi = 2 \text{ rad/sec.} \]

This is the gain cross-over frequency
\[ \omega_0 = 2 \text{ rad/sec.} \]

\[ \frac{G(j\omega)H(j\omega)}{\omega = \omega_0} = \frac{2}{j2(1 + j1)(1 + j0.1)} = 0.707 \angle -140.71^\circ \]

\[ \angle G(j\omega)H(j\omega) = -140.71^\circ \]

According to definition
\[ \text{P.M.} = 180^\circ + \frac{\angle G(j\omega)H(j\omega)}{\omega = \omega_0} = 180^\circ - 140.71^\circ \]
\[ \text{P.M.} = 39.28^\circ \]

From Bode's plot, phase margin and gain margin can be determined as follows.

Draw a line downwards from gain cross-over point '2' till it intersects phase angle plot. This point of intersection is represented by '2' the distance between point '2' and -180° line is the phase margin. If point '2' on phase angle plot is above -180° line, phase margin is positive and if point '2' is below -180° line, phase margin is negative (as shown in fig. 4.29).

From phase cross-over point '2' draw a line upwards till it intersects magnitude plot at point '1'. The difference between magnitude corresponding to point '1' and '0' dB is gain margin. If point '1' is below 0 dB G.M. is positive and if point '1' is above 0 dB line G.M. is negative as shown in fig. 4.29.

For positive G.M. and P.M. the system is stable and the gain cross-over frequency = phase cross-over frequency. If gain cross-over frequency > phase cross-over frequency both P.M. & G.M. will be negative and the system is unstable. If gain cross-over frequency = phase cross-over frequency, the system is marginally stable.

The transfer function changes with the change of temperature & pressure. The gain of the system is also affected by supply voltage, supply frequency, air pressures. If the gain is high, the gain margin is low and for step response the settling time will be large. If the gain is low G.M. & P.M. will be high and settling time will be more i.e., the system response will be sluggish and also the rise time and steady state error will be high.

4.12. DETERMINATION OF K_p, K_a AND K_f FROM BODE PLOT

(a) Static Position Error Constant K_p:

Static position error constant is defined for type zero system and is given by

\[ K_p = \lim_{s \to 0} G(s)H(s) \]

Put \( s = j\omega \)

\[ K_p = \lim_{\omega \to 0} G(j\omega)H(j\omega) \]
From equation (4.4)

\[ 20 \log_{10} \left| G(j\omega) H(j\omega) \right| = 20 \log_{10} K \quad \therefore N = 0 \]

Consider the Bode plot shown in fig. 4.26, the magnitude of \( G(j\omega) \) \( H(j\omega) \) is \( 20 \log K \) which is \( K_p \):

\[ K_p = 20 \log K \]

(b) Static Velocity Error Constant \( K_v \):

Static velocity error constant is defined for type one system and is given by

\[ K_v = \lim_{s \to 0} sG(s)H(s) \]

From equation (4.4) put \( N = 1 \)

\[ 20 \log_{10} \left| G(j\omega) H(j\omega) \right| = 20 \log_{10} K - 20 \log_{10} \omega \]

\[ 0 \text{ db} = 20 \log_{10} K - 20 \log_{10} \omega \]

\[ K = \omega \]

This is static velocity error constant \( K_v = \omega \)

For type one system, the initial slope of Bode plot (magnitude) is \(-20 \text{ db/decade}\). So, draw a line having slope \(-20 \text{ db/dec.} \) which intersect 0 db line. The frequency at which initial slope of a line is \(-20 \text{ db/dec.} \) intersects 0 db line is the value of static velocity error constant \( K_v \) as shown in fig. 4.27.

(c) Acceleration Error Constant \( K_a \):

Acceleration error constant is defined for type two system and is given by

\[ K_a = \lim_{s \to 0} s^2 G(s)H(s) \]

From equation 4.4.

\[ 20 \log_{10} \left| G(j\omega) H(j\omega) \right| = 20 \log_{10} K - 20.2 \log_{10} \omega \]

Extend the line having slope \(-40 \text{ db/dec.} \) to intersect 0 db line.

\[ 0 = 20 \log_{10} K - 40 \log_{10} \omega \]

or,\[ \omega = \sqrt{K} \]

or,\[ K = \omega^2 \]

This is the acceleration error constant.

So, the frequency at which initial line of slope \(-40 \text{ db/decade} \) intersects 0 db line, gives the value of square root of acceleration constant, as shown in fig. 4.27.

\[ \omega = \sqrt{K_a} \]

\[ K_a = \omega^2 \]

4.13. BODE PLOT WITH TRANSPORTATION LAG

Transportation lag is also known as time delay. Due to the various reasons sometimes it is necessary to stop the action for some time. Such time delay is called transportation lag. In many systems, such as hydraulic, pneumatic and thermal pure delay occurs. In Laplace domain the transportation lag is given by \( e^{-Ts} \), where \( T \) is the delay in seconds.

\[ G(s) = e^{-Ts} \]

Put \( s = j\omega \)

\[ G(j\omega) = e^{-j\omega T} \]

we know that

\[ e^{-j\theta} = \cos \theta - j \sin \theta \]

\[ G(j\omega) = \cos \omega T - j \sin \omega T \]

\[ |G(j\omega)| = \sqrt{\cos^2 \omega T + \sin^2 \omega T} = 1 \]

Therefore, the magnitude is always equal to unity for all values of \( \omega \).

In db = \( 20 \log |G(j\omega)| \) = \( 20 \log 1 = 0 \)

In other words we can say that the introduction of time delay has no effect on magnitude plot.

Phase Angle of Transportation Lag

\[ \angle G(j\omega) = \tan^{-1} \left( \frac{-\sin \omega T}{\cos \omega T} \right) \]

\[ = -\tan^{-1}(\tan \omega T) = -\omega T \text{ (radians)} \]

\( \omega \) in radians/sec. and \( t \) in seconds, therefore phase lag is in radians.

\[ \angle G(j\omega) = -57.3 \omega T \text{ degree} \]

The phase angle linearly varies with the frequency \( \omega \).

<table>
<thead>
<tr>
<th>( \omega )</th>
<th>(-57.3 \omega T)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.1</td>
<td>-5.73°</td>
</tr>
<tr>
<td>0.5</td>
<td>-28.65°</td>
</tr>
<tr>
<td>1.0</td>
<td>-114.6°</td>
</tr>
<tr>
<td>5.0</td>
<td>-296.5°</td>
</tr>
<tr>
<td>10</td>
<td>-573°</td>
</tr>
</tbody>
</table>

Example 4.9. Sketch the Bode plot for the transfer function

\[ G(s) = \frac{1000}{(1+0.1s)(1+0.001s)} \]

Determine the a. P.M  
b. Gain margin  
c. Stability of the system.

Solution: Step 1: Put \( s = j\omega \)

\[ G(j\omega) = \frac{1000}{(1+j0.1\omega)(1+j0.001\omega)} \]

The given transfer function is of type '0' system. Therefore the initial slope of the Bode plot is 0 db/decade. The starting point is given by.

\[ 20 \log_{10} K = 20 \log_{10} 1000 = 60 \text{ db} \]

corner frequencies

\[ \omega_1 = \frac{1}{0.1} = 10 \text{ rad/sec.} \]

\[ \omega_2 = \frac{1}{0.001} = 1000 \text{ rad/sec.} \]
Step 2: Mark the starting point 60 db on y-axis and draw a line of slope 0db/decade up to first corner frequency.

Step 3: From first corner frequency to second corner frequency, draw a line with slope (0 - 20) = 20 db/decade.

Step 4: From second corner frequency to next corner frequency (if given) draw a line having negative slope -20 + (-20) = -40 db/decade.

Step 5: The magnitude plot is complete and now draw the phase plot by calculating the phase at different frequencies (as given in Table).

Step 6: From the bode plot, from the point of intersection of magnitude curve with 0 db axis draw a line on phase curve. This line cuts the phase curve at $-154^\circ$.

$P.M = 154 - 180 = +26^\circ$

Step 7: Gain margin $G.M = \infty$.

Since $P.M = +26^\circ$ and gain margin $= \infty$, the system is inherently stable.

<table>
<thead>
<tr>
<th>$\omega$</th>
<th>Arg $(1 + j0.1\omega)$</th>
<th>Arg $(1 + j0.001\omega)$</th>
<th>Resultant</th>
</tr>
</thead>
<tbody>
<tr>
<td>50</td>
<td>$-78.6^\circ$</td>
<td>$-2.86^\circ$</td>
<td>$-81.46^\circ$</td>
</tr>
<tr>
<td>100</td>
<td>$-84.2^\circ$</td>
<td>$-5.7^\circ$</td>
<td>$-90^\circ$</td>
</tr>
<tr>
<td>150</td>
<td>$-86.2^\circ$</td>
<td>$-8.5^\circ$</td>
<td>$-94^\circ$</td>
</tr>
<tr>
<td>200</td>
<td>$-87.13^\circ$</td>
<td>$-11.3^\circ$</td>
<td>$-98^\circ$</td>
</tr>
<tr>
<td>500</td>
<td>$-88.85^\circ$</td>
<td>$-26.56^\circ$</td>
<td>$-115.4^\circ$</td>
</tr>
<tr>
<td>800</td>
<td>$-89.28^\circ$</td>
<td>$-38.65^\circ$</td>
<td>$-127.93^\circ$</td>
</tr>
<tr>
<td>1000</td>
<td>$-89.48^\circ$</td>
<td>$-45^\circ$</td>
<td>$-134.42^\circ$</td>
</tr>
<tr>
<td>2000</td>
<td>$-89.72^\circ$</td>
<td>$-63.43^\circ$</td>
<td>$-153.15^\circ$</td>
</tr>
<tr>
<td>3000</td>
<td>$-89.8^\circ$</td>
<td>$-71.56^\circ$</td>
<td>$-161.36^\circ$</td>
</tr>
<tr>
<td>5000</td>
<td>$-89.68^\circ$</td>
<td>$-78.69^\circ$</td>
<td>$-168.57^\circ$</td>
</tr>
<tr>
<td>8000</td>
<td>$-89.92^\circ$</td>
<td>$-82.87^\circ$</td>
<td>$-172.79^\circ$</td>
</tr>
</tbody>
</table>
Example 4.10. Sketch the Bode plot for the transfer function.

\[ G(s) = \frac{1000}{s(1+0.1s)(1+0.001s)} \]

- (i) Gain crossover frequency
- (ii) Phase crossover frequency
- (iii) G.M & P.M
- (iv) Stability of the given system

Solution: Step 1:
Put \( s = j\omega \)

\[ G(j\omega) = \frac{1000}{(\omega)(1+j0.1\omega)(1+j0.001\omega)} \]

Step 2: Draw the magnitude curve
- Corner frequencies \( \omega_1 = 1/0.1 = 10 \text{ rad/sec} \)
- \( \omega_2 = 1/0.001 = 1000 \text{ rad/sec} \)
- Initial slope of the curve will be \(-20 \text{ dB/decade} \) due to \(1/j\omega \) term. On \( \omega \)-axis mark \( \omega = K \times 100 \).

From this point draw a line having the slope \(-20 \text{ dB/decade} \) to meet \( y \)-axis. This will be the starting point. From the starting point to the first corner frequency the slope will be \(-20 \text{ dB/decade} \). After \( 1000 \), the slope will be \(-20 \text{ dB/decade} \) to \(-60 \text{ dB/decade} \).

Step 3: Draw the phase curve

\[ \phi = -90^\circ - \tan^{-1} 0.1\omega - \tan^{-1} 0.001\omega \]

Table 4.5.

<table>
<thead>
<tr>
<th>( \omega )</th>
<th>( \text{Arg}(j\omega))</th>
<th>( \text{Arg}(1+j0.1\omega))</th>
<th>( \text{Arg}(1+j0.001\omega))</th>
<th>Resultant ( \phi )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>-90°</td>
<td>-5.7°</td>
<td>-0.06°</td>
<td>-95.7°</td>
</tr>
<tr>
<td>5</td>
<td>-90°</td>
<td>-26.5°</td>
<td>-0.28°</td>
<td>-116.5°</td>
</tr>
<tr>
<td>10</td>
<td>-90°</td>
<td>-45°</td>
<td>-0.57°</td>
<td>-135.6°</td>
</tr>
<tr>
<td>50</td>
<td>-90°</td>
<td>-78.6°</td>
<td>-2.86°</td>
<td>-171.4°</td>
</tr>
<tr>
<td>100</td>
<td>-90°</td>
<td>-84.2°</td>
<td>-5.7°</td>
<td>-179.9°</td>
</tr>
<tr>
<td>150</td>
<td>-90°</td>
<td>-86.2°</td>
<td>-8.5°</td>
<td>-184°</td>
</tr>
<tr>
<td>200</td>
<td>-90°</td>
<td>-87.13°</td>
<td>-11.3°</td>
<td>-188°</td>
</tr>
<tr>
<td>500</td>
<td>-90°</td>
<td>-88.85°</td>
<td>-26.56°</td>
<td>-205.4°</td>
</tr>
</tbody>
</table>

Step 1: From Bode plot
- (i) Gain crossover frequency = 100 rad/sec.
- (ii) Phase crossover frequency = 100 rad/sec.
- (iii) G.M = 0
- P.M = 0
- (iv) Since G.M = P.M = 0, the system is marginally stable

![Fig. 4.31](image-url)
Example 4.11. Draw the Bode plot for the transfer function
\[ G(s) = \frac{16(1 + 0.5s)}{s^2(1 + 0.125s)(1 + 0.1s)} \]

From the graph determine
(i) Phase cross over frequency
(ii) Gain cross over frequency
(iii) P.M
(iv) G.M
(v) Stability of the system

Solution: Step 1: Put
\[ s = jo \]
\[ G(\omega) = \frac{16(1 + j0.5\omega)}{(j\omega)^2(1 + 0.125\omega)(1 + j0.1\omega)} \]

Step 2: Draw the magnitude plot
Corner frequencies
\[ \omega_1 = 1/0.5 = 2 \text{ rad/sec} \]
\[ \omega_2 = 1/0.125 = 8 \text{ rad/sec} \]
\[ \omega_3 = 1/0.1 = 10 \text{ rad/sec} \]

At \( \omega = \sqrt{K} = \sqrt{16} = 4 \text{ rad/sec} \) because this is a type two system. From 4 rad/sec, draw a line having the slope of -40 db/decade to meet the y-axis. This will be the starting point. From the starting point to the first corner frequency the slope will be -40 db/decade. From first corner frequency (2 rad/sec) to the second corner frequency (8 rad/sec) the slope of the line will be -20 db/decade. From second corner frequency to the third corner frequency (10 rad/sec) the slope of the line will be -20 + (-20) = -40 db/decade. After 10 rad/sec, the slope will be 40 db/decade.

Step 3: Draw the phase diagram

Step 4: Phase cross over frequency = 6.5 rad/sec \( (\omega_c) \)
Gain cross over frequency = \( \omega_c = 8 \text{ rad/sec} \)
\[ P.M = -8^\circ \]
\[ G.M = -2 \text{ db} \]

(i) Since gain crossover frequency > phase cross over frequency hence the system is unstable. Also both P.M & G.M are negative.

Table 4.6.

<table>
<thead>
<tr>
<th>( \omega )</th>
<th>(-\omega^2)</th>
<th>(-\tan^{-1} 0.125\omega)</th>
<th>(-\tan^{-1} 0.1\omega)</th>
<th>(+ \tan^{-1} 0.5\omega)</th>
<th>Resultant</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.1</td>
<td>-180°</td>
<td>-0.716°</td>
<td>-0.57°</td>
<td>+2.86°</td>
<td>-178.4°</td>
</tr>
<tr>
<td>0.2</td>
<td>-180°</td>
<td>-1.43°</td>
<td>-1.15°</td>
<td>+5.71°</td>
<td>-176.8°</td>
</tr>
<tr>
<td>0.5</td>
<td>-180°</td>
<td>-3.57°</td>
<td>-2.86°</td>
<td>+14°</td>
<td>-172.4°</td>
</tr>
<tr>
<td>0.8</td>
<td>-180°</td>
<td>-5.71°</td>
<td>-4.57°</td>
<td>+23.8°</td>
<td>-168.4°</td>
</tr>
<tr>
<td>1.0</td>
<td>-180°</td>
<td>-7.13°</td>
<td>-5.71°</td>
<td>+26.56°</td>
<td>-166.2°</td>
</tr>
<tr>
<td>2.0</td>
<td>-180°</td>
<td>-14°</td>
<td>-11.3°</td>
<td>+45°</td>
<td>-160.3°</td>
</tr>
<tr>
<td>5.0</td>
<td>-180°</td>
<td>-32°</td>
<td>-26.56°</td>
<td>+68.19°</td>
<td>-170.37°</td>
</tr>
<tr>
<td>8.0</td>
<td>-180°</td>
<td>-45°</td>
<td>-38.66°</td>
<td>+76°</td>
<td>-187°</td>
</tr>
<tr>
<td>10.0</td>
<td>-180°</td>
<td>-51.34°</td>
<td>-45°</td>
<td>+78.69°</td>
<td>-197.65°</td>
</tr>
<tr>
<td>20.0</td>
<td>-180°</td>
<td>-68.19°</td>
<td>-63.43°</td>
<td>+84.28°</td>
<td>-227.34°</td>
</tr>
<tr>
<td>30.0</td>
<td>-180°</td>
<td>-75°</td>
<td>-71.56°</td>
<td>+86.18°</td>
<td>-240.38°</td>
</tr>
</tbody>
</table>

Fig. 4.32.
Example 4.12. Draw the Bode plot for the transfer function

\[ G(s) = \frac{50}{s(1 + 0.25s)(1 + 0.1s)} \]

From the graph determine:
1. Gain crossover frequency
2. Phase crossover frequency
3. G.M & P.M.
4. Stability of the system.

Solution: Since this is type one system, the initial slope of the line will be \(-20\) db/decade. Mark a point on \(s\)-axis at 50 rad/sec and draw a line with slope \(-20\) db/dec. from the point at 50, this line will meet the \(y\)-axis. This is the starting point.

Corner frequencies are:
- \(\omega_1 = \frac{1}{0.25} = 4\) rad/sec.
- \(\omega_2 = \frac{1}{0.1} = 10\) rad/sec.

Slope of the line from starting point to I corner frequency = \(-20\) db/dec.
Slope of the line from I corner frequency to II = \(-20 + (-20)\) = \(-40\) db/dec.

After second corner frequency the slope will be \(-40\) + \((-20)\) = \(-60\) db/dec.

Phase Plot: For phase plot calculate the phases at different frequencies.

<table>
<thead>
<tr>
<th>(\omega)</th>
<th>(\text{j}\omega)</th>
<th>(-\tan^{-1} 0.1\omega)</th>
<th>(-\tan^{-1} 0.25\omega)</th>
<th>Resultant</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>-90°</td>
<td>5.71°</td>
<td>14.03°</td>
<td>109.74°</td>
</tr>
<tr>
<td>2</td>
<td>-90°</td>
<td>11.3°</td>
<td>26.56°</td>
<td>127.86°</td>
</tr>
<tr>
<td>5</td>
<td>-90°</td>
<td>26.56°</td>
<td>51.34°</td>
<td>168°</td>
</tr>
<tr>
<td>10</td>
<td>-90°</td>
<td>45°</td>
<td>68.19°</td>
<td>203.19°</td>
</tr>
<tr>
<td>20</td>
<td>-90°</td>
<td>63.43°</td>
<td>78.69°</td>
<td>232.12°</td>
</tr>
<tr>
<td>40</td>
<td>-90°</td>
<td>76°</td>
<td>84.28°</td>
<td>250.28°</td>
</tr>
<tr>
<td>60</td>
<td>-90°</td>
<td>80.53°</td>
<td>86.18°</td>
<td>256.71°</td>
</tr>
<tr>
<td>80</td>
<td>-90°</td>
<td>82.87°</td>
<td>87.13°</td>
<td>260°</td>
</tr>
<tr>
<td>100</td>
<td>-90°</td>
<td>84.29°</td>
<td>87.7°</td>
<td>262°</td>
</tr>
</tbody>
</table>

Gain crossover frequency \((\omega_{gc}) = 13\) rad/sec.
Phase crossover frequency \((\omega_{pc}) = 6.5\) rad/sec.
P.M. = -36°
G.M. = -13°

Since, both phase margin & gain margin are negative and gain crossover frequency > phase crossover frequency, the system is unstable.
Example 4.13. Draw the Bode plot for

\[ G(s) = \frac{23.7(1 + j\omega)(1 + j0.2\omega)}{(j\omega)(1 + j3\omega)(1 + j0.5\omega)(1 + j0.1\omega)} \]

From the plot find G.M & P.M.

Solution:

Step 1: On \( \omega \)-axis mark the point at 23.7 rad/sec. Since in denominator (j\omega) term is having power one, from 23.7 draw a line of slope -20 db/decade to meet y-axis. This will be the stable point.

From the starting point to the I corner frequency (0.33) the slope of the line will be -20 db/decade.

From I corner frequency (0.33) to second corner frequency (1) the slope of the line will be 20 + (-20) = -20 db/decade.

From II corner frequency (1) to III corner frequency (2) the slope of the line will be -40 + (+20) = -20 db/decade.

From III corner frequency to IV corner frequency, the slope of the line will be -20 + (+20) = -40 db/decade.

From IV corner frequency (5) to V corner frequency, the slope of the line will be -40 + (+20) = -20 db/decade.

After V corner frequency, the slope will be -20 + (-20) = -40 db/decade.

Step 2: Draw the phase plot.

Values of \( \phi \) at different frequencies are tabulated.

Step 3: From graph

P.M = +34°

G.M = \( \infty \)

<table>
<thead>
<tr>
<th>( \omega )</th>
<th>( \tan^{-1}0 )</th>
<th>( \tan^{-1}3\omega )</th>
<th>( \tan^{-1}0.5\omega )</th>
<th>( \tan^{-1}0.1\omega )</th>
<th>( \tan^{-1}0.2\omega )</th>
<th>Result</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.1</td>
<td>-90°</td>
<td>-16.7°</td>
<td>-2.86°</td>
<td>-0.57°</td>
<td>+5.71°</td>
<td>+1.14°</td>
</tr>
<tr>
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<td>-90°</td>
<td>-31°</td>
<td>-5.71°</td>
<td>-1.14°</td>
<td>+11.3°</td>
<td>+2.3°</td>
</tr>
<tr>
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<td>-90°</td>
<td>-56.3°</td>
<td>-14.03°</td>
<td>-2.86°</td>
<td>+26.56°</td>
<td>+5.71°</td>
</tr>
<tr>
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<td>-21.8°</td>
<td>-4.57°</td>
<td>+38.65°</td>
<td>+9.09°</td>
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<td>-26.56°</td>
<td>-5.71°</td>
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<td>+58°</td>
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<td>-84.3°</td>
<td>-63.43°</td>
<td>+87.13°</td>
<td>+76°</td>
</tr>
</tbody>
</table>
Example 4.14. Sketch the Bode plot

\[ G(s) = \frac{512(s + 3)}{s(s^2 + 16s + 256)} \]

Solution: Step 1

\[ G(s) = \frac{512\left(\frac{s}{3} + 1\right)}{s\left(\frac{s^2}{256} + \frac{16s}{256} + 1\right)} \]

\[ = \frac{6(1 + s/3)}{s\left(\frac{s^2}{256} + \frac{16}{256}s + 1\right)} \]

Compare \( \frac{s^2}{256} + \frac{16}{256}s + 1 \) with \( \frac{s^2}{\omega_n^2} + \frac{2\zeta}{\omega_n}s + 1 \)

\[ \omega_n^2 = 256 : \omega_n = 16 \text{ rad/sec.} \]

\[ \frac{2\zeta}{\omega_n} = \frac{16}{256} : \zeta = 0.5 \]

Step 2: On \( \omega \)-axis mark the point at 6. From this point draw a line having slope \(-20\) db/decade, meets the \( \gamma \)-axis. This is the starting point. From the starting point to the corner frequency (3 rad/sec) the slope will be \(-20\) db/decade.

Since \( \omega = 3 \) rad/sec, corresponding to a zero, at this point the slope is increased by \(+20\) db/decade. Hence the slope will be \(-20 + (+20) = 0\) db/decade up to the corner frequency i.e., 16 rad/sec.

At \( \omega = 16 \) rad/sec corresponding to the pair of complex poles the slope must be decreased by \(-40\) db/dec. \((0 - 40 = -40\) db/dec.

Step 3: Draw the phase plot

<table>
<thead>
<tr>
<th>( \omega )</th>
<th>( \phi )</th>
<th>( 3 + j\omega )</th>
<th>( s^2 + 16s + 256 )</th>
<th>Resultant</th>
</tr>
</thead>
<tbody>
<tr>
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<td>+1.9°</td>
<td>-0.36°</td>
<td>-88.46°</td>
</tr>
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<td>-0.72°</td>
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<td>-1.79°</td>
<td>-82.29°</td>
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<td>-3.59°</td>
<td>-75.16°</td>
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<td>-7.23°</td>
<td>-63.53°</td>
</tr>
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<td>-19.1°</td>
<td>-50.1°</td>
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<tr>
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<td>-122.7°</td>
</tr>
<tr>
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<td>-90°</td>
<td>+86.56°</td>
<td>-160°</td>
<td>-163.4°</td>
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<tr>
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<td>-90°</td>
<td>+88.28°</td>
<td>-170.67°</td>
<td>-172.4°</td>
</tr>
</tbody>
</table>

Fig. 4.35.
Example 4.15. The magnitude plot of the open loop transfer function $G(s)$ of a certain system is shown in the fig. 4.36.
(a) Determine $G(s)$ if it is known that the system is of minimum phase type.
(b) Estimate the phase at each of the corner frequencies.

\[
\begin{align*}
|G(i\omega)| & = 100 \text{db} \\
\theta & = -20 \text{db/decade} \\
\theta & = -60 \text{db/decade} \\
\theta & = -80 \text{db/decade}
\end{align*}
\]

Fig. 4.36.

Solution: Since, the system is of minimum phase type, it has no poles or zeros in the right hand side of s-plane.

a. At $\omega = 5$ rad/sec, slope changes to $-20$ db/decade indicating a term $\left(1 + \frac{s}{5}\right)$ in denominator.

b. At $\omega = 40$, slope changes to $-60$ db/decade indicating a term $\left(1 + \frac{s}{40}\right)^2$ in denominator (set slope changes is $-40$ db/decade)

c. At $\omega = 100$, slope changes to $-80$ db/decade, indicating the term $\left(1 + \frac{s}{100}\right)$ in denominator.

d. $20 \log_{10} K = 100$

\[
K = 10^5
\]

\[
G(s) = \frac{10^5}{(1 + 0.2s)(1 + 0.025s)(1 + 0.01s)} \quad \text{Ans.}
\]

At $\omega = 5$

\[
\theta_5 = \left[\tan^{-1} 0.2 	imes 5 + 2(\tan^{-1} 0.025 \times 5) + \tan^{-1} 0.01 \times 5\right] = -62.31^\circ
\]

At $\omega = 40$

\[
\theta_{40} = \left[\tan^{-1} 40 \times 0.2 + 2(\tan^{-1} 0.025) + \tan^{-1} 40 \times 0.01\right] = [-82.87 + 90 + 21.8] = 194.67^\circ
\]

At $\omega = 100$

\[
\theta_{100} = \left[\tan^{-1} 100 \times 0.2 + (\tan^{-1} 100 \times 0.025) + \tan^{-1} 100 \times 0.01\right] = [-87.13 + 136.39 + 45] = 266.53^\circ
\]

Example 4.16. Find the transfer function of the system from the data given on Bode diagram.

\[
\begin{align*}
\text{Fig. 4.37.}
\end{align*}
\]

Solution: First line having a slope of $-20$ db/decade and it is not passing through $\omega = 1$ rad/sec it means there is term $K/s$

\[
At \omega = 1 \text{ rad/sec, of initial part} = 20 \log 1.5 = 3.521
\]

\[
20 \log K = 3.521
\]

\[
K = 1.5
\]

At $\omega = 4 \text{ rad/sec}$. There is a term $\left[1 + \frac{\omega_n}{\omega_0} s + \frac{s^2}{\omega_0^2}\right]^{-1}$ because the slope changes from $-20$ db/decade to $-60$ db/decade and also a peak of $4 \text{ db}$ is given.

\[
\omega_n = 4 \text{ rad/sec.}
\]

Value of $\left[1 + \frac{\omega_n}{\omega_c} s + \frac{s^2}{\omega_c^2}\right]^{-1}$ at $\omega = \omega_n$

Put $\omega = j\omega$

\[
\left[1 + \frac{\omega_n}{\omega_c} j\omega + \frac{(j\omega)^2}{\omega_c^2}\right]^{-1}
\]

\[
= \left[\sqrt{1 - \frac{\omega_n^2}{\omega_c^2}} + \frac{\omega_n^2}{\omega_c^2}\right]^{-1}
\]

\[
= \frac{1}{2\xi}
\]

\[
20 \log \frac{1}{2\xi} = 4
\]

\[
\left[1 + \frac{\omega_n}{\omega_c} s + \frac{s^2}{\omega_c^2}\right]^{-1} = [1 + 0.158 s + 0.0625 s^2]^{-1}
\]
At $\omega = 10$ rad/sec, the slope changes from -60 db/dec. to -40 db/dec. indicating the presence of $1 + \frac{s}{10}$ in numerator.

At $\omega = 20$ rad/sec, the slope changes from -40 db/decade to -60 db/dec. indicating the presence of $1 + \frac{s}{20}$ in denominator.

\[ G(s) = \frac{1.5(1+s/10)}{s(1 + 0.15s + 0.0625s^2)(1 + s/20)} \]
\[ G(s) = \frac{48(s+10)}{s(s^2 + 2.52s + 16)(s+20)} \]

This is the required transfer function.

**Example 4.17.** Find the transfer function of the given Bode diagram.

![Bode diagram](image)

**Solution:**
1. First line having a slope of +20 db/decade. Therefore there is a term $s$ in numerator.
2. At $\omega = 1$, the slope changes to zero; it means there is a term $(1 + \frac{s}{12}) = (s + 1)$ in denominator.
3. At $\omega = 10$, the slope changes to -20 db/decade; this indicates there is a term $(1 + \frac{s}{12})$ in denominator.
4. $20 \log K = 6$

\[ K = 1.99 \approx 2.0 \]

\[ G(s) = \frac{2s}{(s+1)(1+s/10)} \]

\[ G(s) = \frac{20s}{(s+1)(s+10)} \] is the required transfer function.

**4.14. FREQUENCY DOMAIN SPECIFICATIONS**

(a) **Resonant Peak** $M_r$: The maximum value of magnitude is known as resonant peak. $M_r$ magnitude of resonant peak gives the information about the relative stability of the system. A large value of resonant peak implies undesirable transient response.

(b) **Resonant Frequency** $(\omega_r)$: The frequency at which magnitude has maximum value is known as resonant frequency. If $\omega_r$ is large, the time response is fast.

(c) **Bandwidth**: Bandwidth is defined as the range of frequencies in which the magnitude
4.15. CORRELATION BETWEEN TIME AND FREQUENCY RESPONSE

Consider the second order system

\[ C(s) = \frac{\omega_n^2}{s^2 + 2\zeta \omega_n s + \omega_n^2} \tag{4.11} \]

or

\[ C(j\omega) = \frac{\omega_n^2}{(\omega_n - j\zeta \omega_n) + j(2\zeta \omega_n \omega)} \]

Let \( \frac{\omega_n}{\omega} = \xi \)

\[ C(j\omega) = \frac{\omega_n^2}{(1 - \xi^2) + j2\xi \omega n} \]

The magnitude and phase angle characteristic for a certain value \( \xi \) is shown in Fig. 4.42.

The frequency at which \( M \) has maximum value is known as resonant frequency (\( \omega_r \))

\[ \omega_r = \frac{\omega_n}{\sqrt{1 - 2\xi^2}} \] \tag{4.14}

Form equation 4.13.

\[ \theta = -\tan^{-1} \frac{2\xi \omega_n}{1 - \omega_n^2} \]

\[ \omega_n \sqrt{1 - 2\xi^2} \]

\[ \theta = -\tan^{-1} \frac{2\xi \omega_n}{1 - \omega_n^2} \]

\[ \omega_n \sqrt{1 - 2\xi^2} \]

\[ \theta = -\tan^{-1} \frac{2\xi \omega_n}{1 - \omega_n^2} \] \tag{4.15}
For max. $M$ value of magnitude put $u_r = \sqrt{1 - 2\xi^2}$ in eqn 4.12.

$$M_r = \frac{1}{\sqrt{[1-(1-2\xi^2)]^2 + 4\xi^4(1-2\xi^2)}}$$

where $M_r$ is known as resonant peak.

For step response of second order system (for $0 \leq \zeta \leq 1$) maximum overshoot $M_p$ is given by

$$M_p = \frac{\zeta}{\sqrt{1-\zeta^2}}$$

and damped frequency of oscillation is given by

$$\omega_d = \omega_n \sqrt{1-\xi^2}$$

From equation (4.14) and (4.18)

$$\omega_d = \frac{\omega_n \sqrt{1-\xi^2}}{\omega_n \sqrt{1-\xi^2}}$$

From above expression if the value of $\xi$ is small, the damped frequency $\omega_d$ and resonant frequency are nearly same. Hence for large value of $\omega_d$, the time response is faster. 

From equation (4.16) and (4.17) it is clear that both $M_r$ and $M_p$ are the function of $\xi$. As $\xi$ increases both $M_r$ and $M_p$ decreases, but the $M_p$ is limited to 1 and $M_r$ having very large values for $\xi < 0.4$ as shown in Fig. 4.42.

The bandwidth is also defined as the range of frequencies over which $M$ is equal or greater than $\frac{1}{\sqrt{2}}$.

![Diagram](Fig. 4.44)

From equation no. (4.12),

$$M = \frac{1}{\sqrt{(1-u^2)^2 + (2\zeta u)^2}}$$

let $u_b = \text{normalized bandwidth} = \frac{\omega_n}{\omega_p}$ then

$$M = \frac{1}{\sqrt{(1-u_b^2)^2 + (2\zeta u_b)^2}} = \frac{1}{\sqrt{2}}$$

or,

$$1 + u_b^2 + 2u_b^2 + 4\xi^2 u_b^2 = 2$$

or,

$$u_b^4 - 2u_b^2 + 4\xi^2 u_b^2 - 1 = 0$$

or,

$$u_b^4 - 2u_b^2 (1 - 2\xi^2) - 1 = 0$$

Put $u_b^2 = x$

$$x^2 - 2x (1 - 2\xi^2) - 1 = 0$$

$$x = \frac{+2(1-2\xi^2) \pm \sqrt{4(1-2\xi^2)^2 + 4}}{2}$$

$$= 1 - 2\xi^2 \pm \sqrt{1-4\xi^2 + 4^2 + 1}$$

$$= 1 - 2\xi^2 \pm \sqrt{2 - 4\xi^2 + 4\xi^4}$$

consider the positive part

$$u_b^2 = 1 - 2\xi^2 + \sqrt{2 - 4\xi^2 + 4\xi^4}$$

$$u_b = \sqrt{1-2\xi^2 + \sqrt{2 - 4\xi^2 + 4\xi^4}}$$

But

$$u_b = \omega_n \frac{\omega_n}{\omega_p}$$

$$\omega_n = \omega_p \sqrt{1-2\xi^2 + \sqrt{2 - 4\xi^2 + 4\xi^4}}$$

... (4.21)
CALCULATION OF PHASE MARGIN
consider open loop transfer function

\[ G(s) = \frac{\omega_n^2}{s(s + 2\zeta \omega_n)} \]

Put \( s = j\omega \)

\[ G(j\omega) = \frac{\omega_n^2}{j\omega (j\omega + 2\zeta \omega_n)} = -\frac{\omega_n^2}{\omega_n^2 + j2\zeta \omega_n \omega_n^2} \]

\[ |G(j\omega)| = \frac{\omega_n^2}{\sqrt{(-\omega_n^2)^2 + (2\zeta \omega_n \omega_n^2)^2}} \]

At gain cross over frequency \( \omega_c \), \( |G(j\omega)| = 1 \)

\[ \omega_c^2 = \sqrt{(-\omega_n^2)^2 + (2\zeta \omega_n \omega_n^2)^2} \]

\[ \omega_c^4 + 4\zeta^2 \omega_n^2 \omega_c^2 \omega_n^2 - \omega_n^4 = 0 \]

Put \( x = \omega_c^2 \)

\[ x^2 + (4\zeta^2 \omega_n^2) x - \omega_n^4 = 0 \]

\[ x = \frac{-4\zeta^2 \omega_n^2 \pm \sqrt{(4\zeta^2 \omega_n^2)^2 - 4\omega_n^4}}{2} \]

\[ x = -2\zeta^2 \omega_n^2 \pm \omega_n^2 \sqrt{4\zeta^2 + 1} \]

or,

\[ \omega_c^2 = -2\zeta^2 \omega_n^2 \pm \omega_n^2 \sqrt{4\zeta^2 + 1} \]

or,

\[ \omega_c = \omega_n \pm \sqrt{4\zeta^2 + 1} \]

\[ \frac{G(j\omega)}{G(j\omega_n)} = \tan^{-1} \frac{2\zeta \omega_n}{\omega_n - \omega_c} = \tan^{-1} \frac{2\zeta \omega_n}{\omega_n} \]

Put the value of \( \omega_c \)

\[ \phi = \tan^{-1} \frac{2\zeta \omega_n}{\sqrt{4\zeta^2 + 1}} \]

\[ \phi = \frac{2\zeta \omega_n}{\sqrt{4\zeta^2 + 1}} \]

\[ \phi = \tan^{-1} \frac{2\zeta \omega_n}{\sqrt{4\zeta^2 + 1}} \]

or,

\[ \phi = \tan^{-1} \frac{2\zeta \omega_n}{\sqrt{4\zeta^4 + 1}} \]

\[ \tan (180 + \phi) = \tan \phi \]

4.18. RELATIVE AND ABSOLUTE STABILITY

There are two types of stability namely absolute stability and relative stability. Absolute stability means whether the system is stable or not i.e. the system is stable or unstable. If the system is stable then we determine how stable it is i.e. we measure the degree of stability, it is known as relative stability.

In time domain the relative stability is measured by maximum overshoot and damping ratio. In frequency domain relative stability is measured by resonant peak \( M_r \).

Example 4.18. The forward path transfer function of a unity feedback control system is

\[ G(s) = \frac{100}{s(s + 6.54)} \]

Find the resonance peak \( M_r \), resonant frequency \( \omega_n \), and bandwidth of the closed loop system.

Solution: Given that

\[ G(s) = \frac{100}{s(s + 6.54)} \]

\[ H(s) = 1 \]

\[ \frac{C(s)}{R(s)} = \frac{G(s)}{1 + G(s)H(s)} \]

\[ = \frac{100(s + 6.54)}{100 + s + 6.54} \]

\[ = \frac{100}{s(s + 6.54) + 100} \]

\[ = \frac{100}{s^2 + 6.54s + 100} \]

\[ \omega_n^2 = \frac{\omega_n^2}{s^2 + 6.54s + 100} \]

\[ \omega_n^2 = 100 \]

\[ \omega_n = 10 \text{ rad/sec} \]

\[ 2\zeta \omega_n = 6.54 \]

\[ 2\zeta 10 = 6.54 \]

\[ \zeta = 0.0327 \]

\[ \omega_r = \omega_n \sqrt{1 - 2\zeta^2} \]

\[ = 10 \sqrt{1 - 2(0.0327)^2} \]

\[ = 8.86 \text{ rad/sec} \]

\[ M_r = \frac{1}{2\zeta \sqrt{1 - \zeta^2}} \]

\[ = \frac{1}{2(0.0327) \sqrt{1 - (0.0327)^2}} \]

\[ = 1.618 \]

Bandwidth is

\[ \omega_r = \omega_n \sqrt{1 - 2\zeta^2 + (2 - 4\zeta^2 + 4\zeta^4)^{1/2}} \]

\[ = 10 \sqrt{1 - 2(0.0327)^2 + (2 - 4 \times 0.0327^2 + 4 \times 0.0327^4)^{1/2}} \]

\[ = 14.34 \text{ rad/sec} \]
Example 4.19. The specification on a second order unity feedback control system with closed loop transfer function.

\[
\frac{C(s)}{R(s)} = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}
\]

are that the maximum overshoot must not exceed 30%. Find the corresponding limiting values of \(M_p\) and BW.

Solution: we know that

\[M_p = e^{-\frac{x}{\sqrt{1-x^2}}} \times 100\]

or,

\[e^{-\frac{x}{\sqrt{1-x^2}}} = \frac{30}{100} = 0.3\]

\[\frac{\pi x}{\sqrt{1-x^2}} = 1.2\]

(take the natural log)

or,

\[\pi x^2 = 1.2^2 (1 - \xi^2)\]

or, \(\xi = 0.356\)

\[M_r = \frac{1}{2\pi\sqrt{1-\xi^2}}\]

\[M_p = \frac{1}{2 \times 0.356 \sqrt{1-0.356^2}}\]

\[\boxed{M_p = 1.5}\]

Normalized BW

\[= \sqrt{1-2\xi^2 + (2 - 4\xi^2 + 4\xi^4)^{1/2}}\]

\[= \sqrt{1-2 \times 0.356^2 + (2 - 4 \times 0.356^2 + 4 \times 0.356^4)^{1/2}} = 1.41 \text{ rad/sec.}\]

Example 4.20. The closed loop frequency response magnitude versus frequency of a second order system is shown in fig. 4.45. Find \(%M_p\) (max\text{m.} overshoot) and peak time.

Solution: From fig. \(M_p = 1.4\) & \(\omega_n = 3\) rad/sec.

Since,

\[M_r = \frac{1}{2\pi\sqrt{1-\xi^2}}\]

\[\therefore 1.4 = \frac{1}{2\xi\sqrt{1-\xi^2}}\]

\[\xi^2 - (1 - \xi^2) = 0.1275\]

\[\xi^2 - \xi^2 + 0.1275 = 0\]

or \(x^2 - x + 0.1275 = 0\)

Put \(x = \xi\)

\[x^2 - x + 0.1275 = 0\]

consider \(x = 0.1505\) or \(x = 0.8495\)

\[\xi = 0.387\]

\[M_p = e^{-\frac{x}{\sqrt{1-x^2}}} \times 100 = e^{-0.387/\sqrt{1-0.387^2}} \times 100 = 26\%\]

\[\boxed{\text{Ans.}}\]
\[
\frac{C(j\omega)}{R(j\omega)} = \frac{\angle C(j\omega)}{\angle R(j\omega)} = \frac{r}{\theta} = r - \theta
\]

Where \(M(\omega)\) is the magnitude and \(\phi(\omega) = r - \theta\)
Frequency response consists of two parts (1) magnitude (2) phase angle. Both can be plotted against different values of \(\omega\).
The constant magnitude loci and constant phase angle can be plotted in complex plane. From these loci we can obtain the frequency response of closed loop system.

4.18. CONSTANT MAGNITUDE CIRCLE (M – CIRCLE)

Let \(G(j\omega) = x + jy\)

Then from eqn (4.25)
\[
M^2 = \left| \frac{C(j\omega)}{R(j\omega)} \right|^2 = \frac{G(j\omega)}{1 + G(j\omega)}
\]

Squaring both sides
\[
M^2 = \frac{x^2 + y^2}{(1 + x)^2 + y^2}
\]

or
\[
M^2 [1 + x^2 + y^2] = x^2 + y^2
\]

\[
M^2 (x^2 + 2x + 1 + y^2) = x^2 + y^2
\]

\[
x^2(M^2 - 1) + y^2(M^2 - 1) + 2xM^2 + 2M^2y^2 = 0
\]

or
\[
x^2(1 - M^2) + y^2(1 - M^2) - 2xM^2 = 0
\]

Divide both the sides by \((1 - M^2)\)
\[
x^2 - y^2 - 2x \frac{M^2}{1-M^2} = \frac{M^2}{1-M^2}
\]

\[
\left( \frac{M^2}{1-M^2} \right)^2 \text{ add in both sides}
\]

\[
x^2 + y^2 - 2x \frac{M^2}{1-M^2} = \frac{M^2}{1-M^2} \left( \frac{M^2}{1-M^2} \right)^2
\]

\[
\left( \frac{x - \frac{M^2}{1-M^2}}{1-M^2} \right)^2 + \left( \frac{y - 0}{1-M^2} \right)^2 = \frac{M^4}{(1-M^2)^2}
\]

Equation (4) is the equation of the circle with centre \(\left( \frac{M^2}{1-M^2}, 0 \right)\) and radius \(\left( \frac{M}{1-M^2} \right)\)

If \(M = 1\) Then equation (4.26) becomes
\[
(1 + x)^2 + y^2 = x^2 + y^2
\]

\[
x = \frac{1}{2}
\]

This is the equation of the straight line parallel to the y-axis and passing through \((-1/2, 0)\) in the \(G\)-plane.

The constant M loci for different value of \(M\) shown in fig 4.48. From fig 4.48 it is clear that
(a) The loci are symmetrical with respect to \(M = 1\)
(b) The \(M\) circles for \(M > 1\) are on the left side of the line \(M = 1\) and for \(M < 1\) the constant \(M\) circles are on right side of the line \(M = 1\).

The intersection of \(G(j\omega)\) plot (Nyquist plot) & constant \(M\) loci gives the value of magnitude \(M\).
The \(M\)-circle which is tangent to the \(G(j\omega)\) plot will give the value of resonance peak (\(M_1\)) and resonance frequency \(\omega_r\) (as shown in fig. 4.47).

![Fig. 4.47.](image)

### Table 4.10. For construction of M-circle

<table>
<thead>
<tr>
<th>S.No.</th>
<th>M</th>
<th>Centre</th>
<th>Radius</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.3</td>
<td>((0.098, 0))</td>
<td>0.029</td>
</tr>
<tr>
<td>2</td>
<td>0.5</td>
<td>((0.33, 0))</td>
<td>0.666</td>
</tr>
<tr>
<td>3</td>
<td>0.67</td>
<td>((0.814, 0))</td>
<td>1.215</td>
</tr>
<tr>
<td>4</td>
<td>0.766</td>
<td>((1.42, 0))</td>
<td>1.854</td>
</tr>
<tr>
<td>5</td>
<td>0.833</td>
<td>((2.27, 0))</td>
<td>2.72</td>
</tr>
<tr>
<td>6</td>
<td>1.2</td>
<td>((-3.27, 0))</td>
<td>-2.72</td>
</tr>
<tr>
<td>7</td>
<td>1.3</td>
<td>((-2.45, 0))</td>
<td>-1.88</td>
</tr>
<tr>
<td>8</td>
<td>1.5</td>
<td>((-1.8, 0))</td>
<td>-1.2</td>
</tr>
<tr>
<td>9</td>
<td>2.0</td>
<td>((-1.33, 0))</td>
<td>-0.66</td>
</tr>
<tr>
<td>10</td>
<td>3.0</td>
<td>((-1.125, 0))</td>
<td>-0.375</td>
</tr>
</tbody>
</table>

4.19. CONSTANT N-CIRCLES (PHASE ANGLE LOCII):
From eq (4.26), the phase shift can be written as
\[
\phi = \arg(x + jy) - \arg(1 + x + jy)
\]
\[
\phi = \tan^{-1} \frac{y}{x} = \tan^{-1} \frac{\frac{y}{\sqrt{1+x^2}}}{\frac{1}{\sqrt{1+x^2}}}
\]

or,
\[
\tan \phi = \frac{\frac{y}{\sqrt{1+x^2}}}{\frac{1}{\sqrt{1+x^2}}} = \frac{y}{x + \sqrt{1+x^2}}
\]

let \[ N = \frac{y}{x^2 + x + y^2} \]

\[ N \left( x^2 + x + y^2 \right) = y \]

\[ x^2 + x + y^2 - \frac{y}{N} = 0 \]

Add \[ \frac{1}{4} + \frac{1}{(2N)^2} \] to both sides, we get
\[ x^2 + x + y^2 - \frac{y}{N} + \frac{1}{4} + \frac{1}{(2N)^2} = \frac{1}{4} + \frac{1}{(2N)^2} \]

or,
\[ (x + 1/2)^2 + \left( y - \frac{1}{2N} \right)^2 = \frac{1}{4} + \frac{1}{(2N)^2} \]

This equation represents the family of the circles, with centres at \[ \left( -\frac{1}{2}, \frac{1}{2N} \right) \] and radius
\[ \frac{1}{\sqrt{\frac{1}{4} + \frac{1}{(2N)^2}}} \]

centre \[ \left( -\frac{1}{2}, \frac{1}{2N} \right) \]
Radius \[ \frac{1}{\sqrt{\frac{N^2 + 1}{4N^2}}} \]

For different values of \( \phi \), \( N \) circles are shown in fig. 4.49
From the fig. 4.49 it is observed that
(a) The centre is lying always at a distance \( x = -1/2 \) and \( y \) depends upon the phase shift.
(b) All the circles passes through \(-1\) as well as \(0\).

<table>
<thead>
<tr>
<th>( \phi )</th>
<th>( N = \tan \phi )</th>
<th>centre</th>
<th>Radius</th>
</tr>
</thead>
<tbody>
<tr>
<td>30°</td>
<td>0.577</td>
<td>(-0.5, 0.866)</td>
<td>1.0</td>
</tr>
<tr>
<td>45°</td>
<td>1.0</td>
<td>(-0.5, 0.5)</td>
<td>0.7</td>
</tr>
<tr>
<td>60°</td>
<td>1.732</td>
<td>(-0.5, 0.288)</td>
<td>0.577</td>
</tr>
<tr>
<td>15°</td>
<td>0.267</td>
<td>0.5, 1.87</td>
<td>1.9</td>
</tr>
<tr>
<td>-60°</td>
<td>-1.732</td>
<td>-0.5, -0.288</td>
<td>0.577</td>
</tr>
<tr>
<td>-45°</td>
<td>-1.0</td>
<td>-0.5, -0.5</td>
<td>0.7</td>
</tr>
<tr>
<td>-30°</td>
<td>-0.577</td>
<td>-0.5, -0.866</td>
<td>1.0</td>
</tr>
</tbody>
</table>
4.20. GAIN ADJUSTMENT BY M-CIRCLE

Consider the fig. 4.50
From triangle AOC:

\[ \sin \theta = \frac{AC}{OA} \]

Where AC = radius of the circle = \( \frac{M}{1 - M^2} \)

\[ OA = \frac{M^2}{1 - M^2} \]

\[ \sin \theta = \frac{M}{1 - M^2} \]

\[ OB = OA - AB \]

\[ OB = OA - AC \sin \theta \]

\[ OB = \frac{M^2}{1 - M^2} - \frac{M}{1 - M^2} \cdot \frac{1}{M} = -1 \]

Thus, the point B represents the point \(-1 + j0\) in the GH-plane and gives the factor by which the gain can be adjusted to obtain desired \(M\).

**PROCEDURE:**

**Step 1:** Draw the polar plot for \(K = 1\) (or any assumed value)
**Step 2:** Calculate \(\theta\) from \(\sin \theta = \frac{1}{M}\)
**Step 3:** Draw a line from origin at angle \(\theta\) with negative real axis
**Step 4:** Draw a circle with center on negative real axis, which is tangent to the line OD and to the polar plot.
**Step 5:** From the point C draw a line BC perpendicular to the negative real axis.
**Step 6:** Required gain can be obtained as

\[ K = \frac{\text{assumed value of } K}{\text{length } OB} \]

**Example 4.21.** The open loop transfer function with unity feedback is given by

\[ G(s) = \frac{K}{s(s+2)(s+4)} \]

Determine the value of \(K\) so that \(M = 2\).

**Solution:**
**Step 1:** Assume \(K = 1\) and draw the polar plot (fig. 4.51)
**Step 2:** \(\sin \theta = \frac{1}{2}\) \(, \ \theta = 30^\circ\)

Draw a line at \(30^\circ\) with negative real axis

**Step 3:** Draw the circle that is tangent to both polar plot and line OD.
**Step 4:** Draw the line from point C perpendicular to the negative real axis intersects at \((-0.06, 0)\)
**Step 5:** Calculate \(K\)
4.21. CLOSED LOOP FREQUENCY RESPONSE FROM M & N-CIRCLES

Frequency response of closed loop system can be obtained with the help of M and N circles. Consider the following example.

The open loop transfer function with unity feedback system is

\[ G(j\omega) = \frac{10}{j\omega (1 + j/0.2\omega) (1 + j/0.05\omega)} \]

Step 1: Draw the polar plot of the given transfer function.
Step 2: Draw M-circles

Step 3: Draw N-circles

Example 4.22. Consider the unity feedback control system whose open loop transfer function is

\[ G(s) = \frac{a \ s + 1}{s^2} \]

Determine the value of \(a\) so that the phase margin is 45°.

Solution: Given that \( G(s) \ H(s) = \frac{1 + as}{s^2} \)

Put \( s = j\omega \)

\[ G(j\omega) \ H(j\omega) = \frac{1 + j\omega}{(j\omega)^2} \]

We know that

\[ P.M. = 180° + \tan^{-1} a\omega \]

\[ = 180° - [180° + \tan^{-1} a\omega] = \tan^{-1} a\omega \]
P.M. = 45° (given)

45° = \tan^{-1} \phi \omega

taking tan on both sides
\tan 45° = \tan (\tan^{-1} \phi \omega)
\phi \omega = 1
\omega = 1/\phi

This is the gain crossover frequency \( \omega_c \).

At gain crossover frequency
\[ |G(j\omega) H(j\omega)| = 1 \]

\[ |G(j\omega) H(j\omega)| = \frac{\sqrt{1 + \phi^2 \omega^2}}{\omega} = 1 \]

\[ \sqrt{1 + \phi^2 \omega^2} = \omega^2 \]

Put the value of \( \omega \)

\[ \sqrt{1 + \phi^2 \frac{1}{\phi^2}} = \frac{1}{\phi} \]

or,
\[ \sqrt{2} = \frac{1}{\phi} \]

\[ \phi = 0.84089 \quad \text{Ans.} \]

Example 4.23. Draw the Bode plot for

\[ G(s) H(s) = \frac{e^{-0.3s}}{s(1+s)} \]

Solution:

\[ G(s) H(s) = \frac{e^{-0.3s}}{s(1+s)} \]

Put \( s = j\omega \)

\[ G(j\omega) H(j\omega) = \frac{e^{-0.3\omega}}{j\omega(1+j\omega)} \]

Since there is a pole at origin, the initial slope of magnitude plot will -20dB/dec. \( e^{-0.3\omega} \) has no effect on magnitude pole.

The corner frequency \( \omega = 1 \) rad/sec.

From \( \omega = 1 \), the slope of next line will be \(-20 + (-20) = -40\)dB/dec. as there is a simple pole.

PHASE PLOT:

\[ \angle G(j\omega) H(j\omega) = -57.3 \times 3\omega = -17.19\omega \]

\[ \angle G(j\omega) H(j\omega) = 90° - \tan^{-1} \omega = 17.19\omega \]
Table 4.12.

<table>
<thead>
<tr>
<th>( \omega )</th>
<th>( \text{Arg}(j\omega) )</th>
<th>( \text{Arg}(1+j\omega) )</th>
<th>( -17.19 \omega )</th>
<th>( \text{Result} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.1</td>
<td>-90°</td>
<td>-5.7°</td>
<td>-1.719°</td>
<td>-97.41°</td>
</tr>
<tr>
<td>0.5</td>
<td>-90°</td>
<td>26.55°</td>
<td>-8.595°</td>
<td>-125.1°</td>
</tr>
<tr>
<td>1.0</td>
<td>-90°</td>
<td>45°</td>
<td>-17.19°</td>
<td>-152.1°</td>
</tr>
<tr>
<td>2.0</td>
<td>-90°</td>
<td>63.43°</td>
<td>-34.38°</td>
<td>-187.9°</td>
</tr>
<tr>
<td>4.0</td>
<td>-90°</td>
<td>75.96°</td>
<td>-68.76°</td>
<td>-234.7°</td>
</tr>
<tr>
<td>5.0</td>
<td>-90°</td>
<td>78.69°</td>
<td>-85.95°</td>
<td>-254.6°</td>
</tr>
<tr>
<td>8.0</td>
<td>-90°</td>
<td>84.28°</td>
<td>-121.9°</td>
<td>-346.1°</td>
</tr>
</tbody>
</table>

The magnitude plot and phase plot are shown in fig. 4.55. From Bode plot:

- Gain crossover frequency \( \omega_c1 = 1 \text{ rad/sec} \).
- Phase crossover frequency \( \omega_c2 = 1.8 \text{ rad/sec} \).
- Gain margin = +10 db
- Phase margin = +28°

**SUMMARY**

1. The magnitude and phase relationship between sinusoidal input and steady state output is a system is known as frequency response.
2. The polar plot of a sinusoidal transfer function \( G(j\omega) \) is a plot of the magnitude of \( G(j\omega) \) versus the phase angle of \( G(j\omega) \) on polar coordinates as \( \omega \) varied from zero to infinity.
3. The phase margin (in polar plot) is that amount of additional phase lag at the gain crossover frequency required to bring the system to the verge of instability.
4. The gain margin is the reciprocal of the magnitude \( |G(j\omega)| \) at the frequency at which the phase angle is -180°.
5. The inverse polar plot at \( G(j\omega) \) is a graph of \( 1/G(j\omega) \) as a function of \( \omega \). Bode plot is a graphical representation of the transfer function for determining the stability of the control system. Bode plot consists of two plots. One is a plot of the logarithm of the magnitude of sinusoidal transfer function, the other is a plot of the phase angle, both plots are plotted against the frequency.
6. The transfer function having no poles and zeros in the right half s-plane are called minimum phase transfer function. Systems with minimum phase transfer function are called minimum phase systems.
7. The transfer function having poles and/or zeros in the right half s-plane are called non-minimum phase transfer functions. Systems with non-minimum phase transfer function are called non-minimum phase systems.
8. In Bode plot the relative stability of the system is determined from the gain margin and phase margin.
9. If the gain crossover frequency is less than phase crossover frequency then gain margin and phase margin both are positive and system is stable.

**EXERCISE**

10. If gain cross over frequency is greater than the phase crossover frequency than both gain margin and phase margin are negative and the system is unstable.
11. If gain cross over frequency is equal to the phase cross-over frequency the gain margin and phase margin are zero and the system is marginally stable.
12. The maximum value of magnitude is known as resonant peak. The magnitude of resonant peak gives the information about the relative stability of the system.
13. The frequency at which magnitude has maximum value is known as resonant frequency.
14. Bandwidth is defined as the range of frequencies in which the magnitude of close loop does not drop 3-dB.
15. The frequency at which the magnitude is 3dB below its zero frequency value is called cut off frequency.

\[
\omega_c = \omega_0 \sqrt{1 - 2\zeta^2}
\]

\[
M_r = \frac{1}{2\zeta \sqrt{1 - \zeta^2}}
\]

16. There are two types of stability namely absolute stability and relative stability. Absolute stability means whether the system is stable or not. If the system is stable then we measure the degree of stability. It is known as relative stability.

41. Sketch the polar plots for the transfer functions

(a) \( G(s) = \frac{1}{s(1+s)(1+2s)} \)
(b) \( G(s) = \frac{10}{s(s+1)(s+4)} \)

42. Sketch the Bode plots for the following transfer function

(a) \( \frac{20s}{(s+1)(s+10)} \)
(b) \( \frac{1000}{s(1+0.1s)(1+0.005s)} \)
(c) \( \frac{1500}{s(s+2)(s+30)} \)

43. Determine the resonant frequency, resonance peak and bandwidth for the system whose transfer function is

\[
\frac{C(s)}{R(s)} = \frac{5}{s^2 + 2s + 5}
\]

44. The forward path transfer function of a unity feedback control system is

\[
G(s) = \frac{21.39}{s(s+6.54)}
\]

Determine the resonance peak, resonant frequency and normalized bandwidth.

45. Repeat the problem 4.19 for maximum overshoot 20%.
SEMIOBJECTIVE TYPE QUESTIONS

(i) Define frequency response.
(ii) What is a polar plot.
(iii) Write the procedure to sketch the polar plot.
(iv) Define minimum phase systems and non-minimum phase systems.
(v) What is a bode plot?
(vi) Define resonant peak, resonant frequency and bandwidth.
(vii) Derive the expression for resonant frequency.
(viii) Define relative and absolute stability.
(ix) Construct the constant N-circle.
(x) Define Phase margin and gain margin.
(xi) Write a short note on correlation between time domain and frequency domain specifications.
(xii) Write a short note on frequency domain specifications.
(xiii) With a suitable diagram define (a) Phase crossover frequency, (b) gain crossover frequency.
(xiv) P.M. (iv) gain margin.
(xv) What information you get from Bode's plot.
(xvi) Why logarithmic scale is used while plotting the Bode plots?
(xvii) State the advantages of Bode plots.
(xviii) State the centre and radius of constant M-circle.
(xix) What is bandwidth?
(xx) What is corner frequency?

Chapter 5

Stability Theory

5.1 CONCEPT OF STABILITY

The concept of stability is very important to analyse and design the system. A system is said to be stable if its response cannot be made to increase indefinitely by the application of a bounded input excitation. If the output approaches towards infinite value for sufficiently large time, the system is said to be unstable.

A linear time invariant (LTI) system is stable if
1. the system is excited by a bounded input, the output is bounded. (BIBO stability criteria)
2. in the absence of the input, the output tends towards zero (the equilibrium state of the system).

This is known as asymptotic stability.

Consider the transfer function

\[
\frac{C(s)}{R(s)} = \frac{a_0 s^n + a_{n-1} s^{n-1} + \ldots + a_1 s + a_0}{b_n s^n + b_{n-1} s^{n-1} + \ldots + b_1 s + b_0}
\]

\( C(t) = \int g(t) r(t - \tau) d\tau \)

where \( g(t) = \mathcal{L}^{-1} G(s) = \text{impulse response of the system} \). So, a system is said to be stable if the impulse response approaches zero for sufficiently large time. If the impulse response approaches infinity for sufficiently large time, the system is said to be unstable. If the impulse response approaches a constant value for sufficiently large time, the system is said to be marginally stable.

5.2 EFFECT OF LOCATION OF POLES ON STABILITY

(a) poles on negative real axis

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{fig51.png}
\caption{Fig. 5.1}
\end{figure}

Consider a simple pole at \( s = -\alpha \) as shown in fig. 5.1a, the corresponding impulse response is given by
(b) Pole on positive real axis

Consider a system having simple pole on positive real axis at \( s = a \), the corresponding impulse response is given by:

\[
g(t) = \mathcal{L}^{-1} \frac{K}{s-a} = Ke^{at}
\]

The response increases exponentially with time, hence the system is unstable. The simple pole and response are shown in fig. 5.2(a) and 5.2(b).

(c) Pole at the origin: Consider a pole at origin

\[
g(t) = \mathcal{L}^{-1} \frac{K}{s} = K
\]

This is constant value, hence the system is marginally stable. If there are two poles at the origin, the time response would be

\[
g(t) = \mathcal{L}^{-1} \frac{K}{s^2} = Kt
\]

(a) Complex poles in the right half of \( s \)-plane

Suppose the system has complex conjugate poles at \( s = \alpha \pm j\omega \). The time response is given by

\[
g(t) = \mathcal{L}^{-1} \left[ \frac{K}{(s-\alpha + j\omega)(s-\alpha - j\omega)} \right] = \mathcal{L}^{-1} \left[ \frac{2A(s-\alpha)}{(s-\alpha)^2 + \omega^2} \right] = 2A e^{\alpha t} \cos \omega t
\]

Hence, the response increases exponentially sinusoidal with time and therefore the response is unstable. The poles and time response shown in fig. 5.6(a) and 5.6(b) respectively.

(f) Poles on \( j\omega \)-axis

If the system having the complex poles on \( j\omega \)-axis the corresponding time response would be

\[
g(t) = \mathcal{L}^{-1} \left[ \frac{A}{s + j\omega} \right] = \mathcal{L}^{-1} \left[ \frac{2As}{s^2 + \omega^2} \right] = 2A \cos \omega t
\]
The response is marginally stable. The equation (5.9) shows the sustained oscillations of some amplitude. This situation will also be considered unstable.

\[ \text{Fig. 5.7.} \]

The overall transfer function is given by

\[ \frac{C(s)}{R(s)} = \frac{G(s)}{1 + G(s)H(s)} \]

The characteristic equation is \(1 + G(s)H(s) = 0\).

The necessary and sufficient condition that a feedback system be stable is that all the roots of the characteristic equation \(1 + G(s)H(s) = 0\) have negative real part. or, in terms of poles we can state that the necessary and sufficient condition that a feedback system be stable is that all the poles of the overall transfer function have negative real part.

### 5.3. NECESSARY BUT NOT SUFFICIENT CONDITIONS FOR STABILITY

Consider a system with characteristic equation

\[ a_0 s^n + a_1 s^{n-1} + \ldots + a_n = 0 \]

(a) All the coefficients of the equation should have same sign,
(b) There should be no missing term.

If above two conditions are not satisfied the system will be unstable. But if all the coefficients have same sign and there is no missing term we have no guarantee that the system will be stable for stability we use ROUTH-HURWITZ CRITERION.

### 5.4. THE ROUTH-HURWITZ CRITERION

Consider the following characteristic polynomial

\[ a_0 s^n + a_1 s^{n-1} + \ldots + a_n = 0 \]

where the coefficients \(a_0, a_1, \ldots, a_n\) are all of the same sign and none is zero.

**Step 1:** Arrange all the coefficients of \(a_0 s^n\) in two rows

<table>
<thead>
<tr>
<th>Row 1</th>
<th>(a_0)</th>
<th>(a_1)</th>
<th>(a_2)</th>
<th>(a_3)</th>
<th>(a_4)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Row 2</td>
<td>(a_1)</td>
<td>(a_2)</td>
<td>(a_3)</td>
<td>(a_4)</td>
<td>(a_5)</td>
</tr>
</tbody>
</table>

**Step 2:** From these two rows form a third row

<table>
<thead>
<tr>
<th>Row 1</th>
<th>(a_0)</th>
<th>(a_1)</th>
<th>(a_2)</th>
<th>(a_3)</th>
<th>(a_4)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Row 2</td>
<td>(a_1)</td>
<td>(a_2)</td>
<td>(a_3)</td>
<td>(a_4)</td>
<td>(a_5)</td>
</tr>
</tbody>
</table>

**Step 3:** From second and third row, form a fourth row

<table>
<thead>
<tr>
<th>Row 1</th>
<th>(a_0)</th>
<th>(a_1)</th>
<th>(a_2)</th>
<th>(a_3)</th>
<th>(a_4)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Row 2</td>
<td>(b_0)</td>
<td>(b_1)</td>
<td>(b_2)</td>
<td>(b_3)</td>
<td>(b_4)</td>
</tr>
</tbody>
</table>

where,

\[ b_1 = \frac{1}{a_1} \begin{bmatrix} a_0 & a_2 \\ a_1 & a_3 \end{bmatrix} \]

\[ b_2 = \frac{1}{a_1} \begin{bmatrix} a_0 & a_1 \\ a_1 & a_2 \end{bmatrix} \]

**Step 4:** Continue this procedure of forming a new rows

#### 5.4.1. Statement of Routh-Hurwitz Criterion

Routh-Hurwitz Criterion states that the system is stable if and only if all the elements in the first column have the same algebraic sign. If all elements are not of the same sign then the number of sign changes of the elements in first column equals the number of roots of the characteristic equation in the right half of the s-plane (or equals to the number of roots with positive real parts).

**Example 5.1.** Check the stability of the system whose characteristic equation is given by

\[ s^3 + 2s^2 + 6s^2 + 4s + 1 = 0 \]

**Solution:** Obtain the array of coefficients as follows

| \(s^3\) | 1 | 6 | 1 |
| \(s^2\) | 2 | 4 |
| \(s^1\) | 3.5 |
| \(s^0\) | 1 |

\[ b_1 = \frac{1}{2} \begin{bmatrix} 1 & 6 \\ 2 & 4 \end{bmatrix} = 2.5 \]

\[ b_2 = \frac{1}{2} \begin{bmatrix} 1 & 1 \\ 2 & 0 \end{bmatrix} = 0.5 \]

Since, all the coefficients in the first column are of the same sign (positive), the given equation has no roots with positive real parts. Hence, the system is stable.

**Example 5.2.** Determine the stability of the system whose characteristic equation is given by

\[ s^3 + 2s^2 + s^3 + 2s + 1 = 0 \]

**Solution:**

| \(s^3\) | 2 | 1 |
| \(s^2\) | 2 | 3 |
| \(s^1\) | 5 |
| \(s^0\) | 2 |

\[ b_1 = \frac{1}{2} \begin{bmatrix} 2 & 1 \\ 2 & 3 \end{bmatrix} = -2 \]

\[ c_1 = \frac{1}{(-2)} \begin{bmatrix} 2 & 3 \\ 2 \end{bmatrix} = 3 \]
Example 5.3. Determine the stability of the system having following characteristic equation
\[ 2s^4 + 5s^3 + 5s^2 + 2s + 1 = 0 \]
Solution:

<table>
<thead>
<tr>
<th>s^4</th>
<th>2</th>
<th>5</th>
<th>1</th>
</tr>
</thead>
<tbody>
<tr>
<td>s^3</td>
<td>5</td>
<td>2</td>
<td></td>
</tr>
<tr>
<td>s^2</td>
<td>4.2</td>
<td>1</td>
<td></td>
</tr>
<tr>
<td>s^1</td>
<td>0.809</td>
<td></td>
<td></td>
</tr>
<tr>
<td>s^0</td>
<td>1</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

From the above Routh table:
- No. of sign changes in first column = 0
- No. of roots on the right hand side of s-plane = 0

Hence, the system is stable.

Example 5.4. Check the stability of the system having following characteristic equation
\[ s^4 + 2s^3 + 3s^2 + 4s + 5 = 0 \]
Solution:

<table>
<thead>
<tr>
<th>s^4</th>
<th>1</th>
<th>3</th>
<th>5</th>
</tr>
</thead>
<tbody>
<tr>
<td>s^3</td>
<td>2</td>
<td>4</td>
<td></td>
</tr>
<tr>
<td>s^2</td>
<td>1</td>
<td>5</td>
<td></td>
</tr>
<tr>
<td>s^1</td>
<td>-6</td>
<td></td>
<td></td>
</tr>
<tr>
<td>s^0</td>
<td>5</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

From above table:
- No. of sign changes in first column = 0
- No. of roots in right half of s-plane = 0

Hence, the system is stable.

Example 5.5. A closed loop control system has the characteristic equation given by
\[ s^3 + 4.5s^2 + 3.5s + 1.5 = 0 \]
Investigate the stability using Routh Hurwitz criterion.

The Routh Hurwitz criterion is also applicable to this equation.

<table>
<thead>
<tr>
<th>s^3</th>
<th>1</th>
</tr>
</thead>
<tbody>
<tr>
<td>s^2</td>
<td>4.5</td>
</tr>
<tr>
<td>s^1</td>
<td>3.17</td>
</tr>
<tr>
<td>s^0</td>
<td>1.5</td>
</tr>
</tbody>
</table>

No. of sign changes in first column = 0
No. of roots in right half of s-plane = 0

Hence, system is stable.

Example 5.6. Check the stability of the system, having following characteristic equation.
\[ s^4 + 6s^3 + 3s^2 + 2s + 1 = 0 \]
Solution:

<table>
<thead>
<tr>
<th>s^4</th>
<th>1</th>
<th>3</th>
</tr>
</thead>
<tbody>
<tr>
<td>s^3</td>
<td>6</td>
<td></td>
</tr>
<tr>
<td>s^2</td>
<td>2.67</td>
<td></td>
</tr>
<tr>
<td>s^1</td>
<td>0.135</td>
<td></td>
</tr>
<tr>
<td>s^0</td>
<td>-2.195</td>
<td></td>
</tr>
</tbody>
</table>

No. of sign change in first column = 2
No. of poles on right half of s-plane = 2

Hence, system is unstable.

SPECIAL CASES

Case 1: If a first column term in any row is zero, but the remaining terms are not zero or there is no remaining term, then multiply the original equation by a factor \((s + a)\) where \(a\) is any positive real number. The simplest value of \(a\) is 1 (take \(a = 1\)). Consider the following example.

Example 5.7. Investigate the stability
\[ s^4 + s^3 + 2s^2 + 3s + 5 = 0 \]
Solution:

<table>
<thead>
<tr>
<th>s^4</th>
<th>1</th>
<th>2</th>
</tr>
</thead>
<tbody>
<tr>
<td>s^3</td>
<td>1</td>
<td></td>
</tr>
<tr>
<td>s^2</td>
<td>5</td>
<td></td>
</tr>
<tr>
<td>s^1</td>
<td>0</td>
<td></td>
</tr>
<tr>
<td>s^0</td>
<td>5</td>
<td></td>
</tr>
</tbody>
</table>

Now, multiply the characteristic equation by \((s + 1)\)
\[(s + 1)(s^4 + s^3 + 2s^2 + 3s + 5) = 0\]
or,
\[s^4 + 2s^3 + 3s^2 + 4s^1 + 5s^0 + 8s + 5 = 0\]

<table>
<thead>
<tr>
<th>s^4</th>
<th>1</th>
<th>3</th>
<th>5</th>
</tr>
</thead>
<tbody>
<tr>
<td>s^3</td>
<td>2</td>
<td></td>
<td></td>
</tr>
<tr>
<td>s^2</td>
<td>1</td>
<td></td>
<td></td>
</tr>
<tr>
<td>s^1</td>
<td>2</td>
<td></td>
<td></td>
</tr>
<tr>
<td>s^0</td>
<td>1</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

From the above table:
- No. of sign change in the first column = 2
- No. of roots in the right half of s-plane = 2

Hence, system is unstable.

Example 5.8. Apply Routh Hurwitz criterion to the following equation and investigate the stability.
\[ s^6 + 2s^5 + 2s^4 + 4s^3 + 11s^2 + 10s + 0 = 0 \]
Solution:

<table>
<thead>
<tr>
<th>s^6</th>
<th>1</th>
</tr>
</thead>
<tbody>
<tr>
<td>s^5</td>
<td>2</td>
</tr>
<tr>
<td>s^4</td>
<td>11</td>
</tr>
<tr>
<td>s^3</td>
<td>4</td>
</tr>
<tr>
<td>s^2</td>
<td>10</td>
</tr>
<tr>
<td>s^1</td>
<td>0</td>
</tr>
<tr>
<td>s^0</td>
<td>0</td>
</tr>
</tbody>
</table>

Since, the third element in first column is zero, so multiply the equation by \((s + 1)\)
\[(s + 1)(s^6 + 2s^5 + 2s^4 + 4s^3 + 11s^2 + 10s + 0) = 0\]

or,
\[s^6 + 3s^5 + 6s^4 + 15s^3 + 21s^2 + 10s + 0 = 0\]
From the above table it is clear that there are two changes of sign in first column, therefore there are two roots in the right half of s-plane. The system is unstable.

**ALTERNATIVE METHOD**

Replacing the zero by a small positive quantity $\epsilon$ and continue the procedure. Consider the following example.

**Example 5.9.** Consider the following equation

$$s^3 + s + 2 = 0$$

**Solution:**

<table>
<thead>
<tr>
<th>$s^3$</th>
<th>1</th>
</tr>
</thead>
<tbody>
<tr>
<td>$s^2$</td>
<td>1</td>
</tr>
<tr>
<td>$s^1$</td>
<td>0</td>
</tr>
<tr>
<td>$s^0$</td>
<td>2</td>
</tr>
</tbody>
</table>

Replace 0 by $\epsilon$ and continue the procedure.

<table>
<thead>
<tr>
<th>$s^3$</th>
<th>1</th>
</tr>
</thead>
<tbody>
<tr>
<td>$s^2$</td>
<td>1</td>
</tr>
<tr>
<td>$s^1$</td>
<td>$\epsilon$</td>
</tr>
<tr>
<td>$s^0$</td>
<td>2</td>
</tr>
</tbody>
</table>

No. of sign changes in first column = 2

No. of roots in right half of s-plane = 2

**Case 2:** When any one row of Routh table is zero.

When any one row of Routh table is zero, it indicates that the equation has at least one pair of roots which lie radially opposite each other and equidistant from origin. The array can be completed by forming the auxiliary polynomial. The polynomial whose coefficients are the element of the row just above the row of zeros in Routh array is called an auxiliary polynomial. Consider the following example.

**Example 5.10.** Consider the equation

$$s^4 + 2s^3 + 4s^2 + 8s + 1 = 0$$

**Solution:**

<table>
<thead>
<tr>
<th>$s^4$</th>
<th>1</th>
</tr>
</thead>
<tbody>
<tr>
<td>$s^3$</td>
<td>2</td>
</tr>
<tr>
<td>$s^2$</td>
<td>0</td>
</tr>
</tbody>
</table>

From above table it is clear that the third row is zero. Hence form the Auxiliary polynomial $A(s)$

$$A(s) = 2s^3 + 4s^2 + 8s + 1$$

$$\frac{dA(s)}{ds} = 6s^2 + 8s + 16$$

Now the Routh array can be written as

<table>
<thead>
<tr>
<th>$s^4$</th>
<th>1</th>
</tr>
</thead>
<tbody>
<tr>
<td>$s^3$</td>
<td>2</td>
</tr>
<tr>
<td>$s^2$</td>
<td>8</td>
</tr>
<tr>
<td>$s^1$</td>
<td>12</td>
</tr>
<tr>
<td>$s^0$</td>
<td>1</td>
</tr>
</tbody>
</table>

No of sign changes in first column = 1

No. of roots in right half of s-plane = 1

The roots of the equation formed by the auxiliary polynomial

$$2s^3 + 4s^2 + 8s + 1 = 0$$

are also the roots of the original equation.

The given equation can be written in factored form as

$$(s+1)(s^3 + 1) = 0$$

It is clear that the original equation has one root with positive real part.

The roots of the auxiliary equation are also the roots of the characteristic equation because auxiliary equation is the part of characteristic equation. Suppose, original characteristic equation is of order six, i.e., having six roots and auxiliary equation is of order four, i.e., having four roots. These four roots are also the roots of the original characteristic equation and remaining ($6 - 4 = 2$) two roots will always lie to the left half of s-plane and these two roots do not have significant role for stability because they lie far away from imaginary axis. The roots of the auxiliary equations are dominant roots, the stability can also be determined from the roots of the auxiliary equation.

Consider the following characteristic equation

$$s^4 + 2s^3 + 8s^2 + 12s + 20 = 0$$

<table>
<thead>
<tr>
<th>$s^4$</th>
<th>1</th>
</tr>
</thead>
<tbody>
<tr>
<td>$s^3$</td>
<td>2</td>
</tr>
<tr>
<td>$s^2$</td>
<td>0</td>
</tr>
</tbody>
</table>

Auxiliary equation $A(s) = 2s^3 + 12s^2 + 16 = 0$. $\frac{dA(s)}{ds} = 8s^2 + 24s$. Since, the order of the characteristic equation is six and order of auxiliary equation is four, the two roots ($6 - 4 = 2$) lie to the left half of s-plane and four roots of auxiliary equations are dominant roots.

<table>
<thead>
<tr>
<th>$s^4$</th>
<th>1</th>
</tr>
</thead>
<tbody>
<tr>
<td>$s^3$</td>
<td>2</td>
</tr>
<tr>
<td>$s^2$</td>
<td>8</td>
</tr>
<tr>
<td>$s^1$</td>
<td>6</td>
</tr>
</tbody>
</table>

No sign change in first column, the system may be stable, since there is a row of zeros it means the roots are lying on imaginary axis. So solve the auxiliary equation $A(s) = 0$.

$$2s^3 + 12s^2 + 16 = 0$$

* Dominant roots means the roots are close to the imaginary axis or on the imaginary axis or they are in right half of s-plane.
5.5 APPLICATION OF ROUTH'S STABILITY CRITERION TO CONTROL SYSTEM ANALYSIS

Routh stability criterion is also used for the determination of stability of the linear feedback systems. Consider the following example.

Example 5.11. The open loop transfer function of unity feedback system is

\[
K
\]

\[
\frac{K}{s(1 + 0.4s)(1 + 0.25s)}
\]

Find the restriction of \( K \) so that the closed loop system is absolutely stable.

Solution: Given that

\[
G(s) = \frac{K}{s(1 + 0.4s)(1 + 0.25s)}
\]

The characteristic equation

\[
1 + G(s)H(s) = 0
\]

or,

\[
s(1 + 0.4s)(1 + 0.25s) + K = 0
\]

or,

\[
s^3 + 6.5s^2 + 10s + 10K = 0
\]

\[
s^3 + 6.5s^2 + 10s + 10K = 0
\]

For absolute stability, there should be no sign change in the first column i.e. no root of the characteristic equation should lie in the right half of \( s \)-plane. This is possible only when

\[K > 0 \text{ and } 65 - 10K > 0\]

Hence, for closed loop stability

\[0 < K < 6.5\]
For stability there should not be any sign change in first column:

\[ a_1 > 0, \ a_2 - a_1 a_3 > 0, \ a_3 a_4 - a_2^2 > 0, \ a_4 > 0 \]

Above are the required conditions.

Example 5.14. The characteristic equations for certain feedback control system are given below. In each case, determine the range of values K for the system to be stable.

\(a.\) \[ s^3 + 20Ks^2 + 5s^2 + 10s + 15 = 0 \]
\(b.\) \[ s^3 + 2Ks^2 + (K + 2)s + 4 = 0 \]

Solution: a. Routh array is:

| \(s^3\) | 1 | 5 | 15 |
| \(20K\) | 10 |
| \(100K - 10\) | 15 |
| \(20K\) | |
| \(600K^2\) | 15 |
| \(10K - 1\) | |

For stability no root should lie in right half of s-plane.

\(a.\) \(20K > 0, \ ie K > 0\)
\(b.\) \(100K - 10 > 0, \ ie K > \frac{1}{2}\)
\(c.\) \(5 > \frac{1}{2K}, \ ie K > \frac{1}{10}\)

For stability no root should lie in right half of s-plane.

\(a.\) \(20K > 0, \ ie K > 0\)
\(b.\) \(100K - 10 > 0, \ ie K > \frac{1}{10}\)
\(c.\) \(5 > \frac{1}{2K}, \ ie K > \frac{1}{10}\)

This gives the complex roots, hence the system is unstable.

(b) \[ s^3 + 2Ks^2 + (K + 2)s + 4 = 0 \]

| \(s^3\) | 1 |
| \(2(K + 2)\) | 4 |
| \(K^2 + 2K - 2\) | |
| \(K\) | |

For stability:

\(2K > 0, \ ie K > 0\)
\(K^2 + 2K - 2 > 0, \ ie K > \frac{3}{4}\)

Example 5.15. A system oscillates with frequency \(\omega\), if it has poles at \(s = \pm j\omega\) and no poles in the right half s-plane. Determine the values of \(K\) and \(a\) so that the system shown in fig. 5.9 oscillates at a frequency of 2 rad/sec.

Solution: The characteristic equation

\[ 1 + G(s) H(s) = 0 \]

\[ \frac{K(s + 1)}{s^3 + 8s^2 + 2s + 1} \]

fig 5.9.

Example 5.16. The open loop transfer function of a unity feedback control system is given by

\[ G(s) = \frac{K}{(s + 2)(s + 4)(s^2 + 8s + 25)} \]

By applying Routh criterion, discuss the stability of the closed loop system as a function of \(K\). Determine the value of \(K\) which will cause sustained oscillations in the closed loop system. What are the corresponding oscillation frequencies?
Solution: The characteristic equation is given by
\[ 1 + G(s)H(s) = 0 \]
\[ \frac{1 + \frac{K}{s(s + 3)(s + 4)(s^2 + 6s + 25)}}{s^2 + 12s^2 + 69s + 198s + (200 + K)} = 0 \]
\[ s^3 = \frac{1}{69} \]
\[ s^2 = 12 \]
\[ s = 198 \]
\[ 200 + K \]

System will be stable when
\[ 200 + K > 0 \quad \text{or} \quad K > -200 \]
\[ 198 - \frac{12(200 + K)}{52.5} > 0 \quad \text{or} \quad K < 666.25 \]

Oscillations will occur when \( K = 666.25 \)
Auxiliary equation: \( 52.5s^2 + (200 + K) = 0 \)
\[ 52.5s^2 + (200 + 666.25) = 0 \]
\[ 52.5s^2 = -866.25 \]
\[ s^2 = -16.5 \]
\[ s = \pm j4.06 \text{ or } \frac{1}{20} = \pm j4.06 \]

:. Frequency of sustained oscillation = 4.06 rad/sec. Ans.

Example 5.17. A unity feedback control system is characterized by open loop transfer function
\[ G(s) = \frac{K(s + 13)}{s(s + 3)(s + 7)} \]

(a) Using Routh criterion, calculate the range of values of \( K \) for the system to be stable.
(b) Check if for \( K = 1 \), all these roots of the characteristic equation of the above system have damping factor greater than 0.5

Solution: The characteristic equation is given by
\[ 1 + G(s)H(s) = 0 \]
\[ \frac{1 + \frac{K(s + 13)}{s(s + 3)(s + 7)}}{s^2 + 10s^2 + 21s + 13K} = 0 \]
\[ s^3 = 1 \]
\[ s^2 = 21 \]
\[ s^1 = 13K \]
\[ s^0 = \frac{3K}{10} \]

\[ \text{For stability:} \quad \begin{cases} 13K > 0 & \text{or} & K > 0 \\ \frac{3K}{10} > 0 & \text{or} & K < 70 \end{cases} \]
\[ \therefore \text{Range is} \quad 0 < K < 70 \]

Example 5.18. A feedback system has an open loop transfer function
\[ G(s)H(s) = \frac{Ke^{-s}}{s(s^2 + 5s + 9)} \]

Determine by use of Routh criterion, the maximum value of \( K \) for the closed loop system to be stable.

Solution: For low frequencies \( s^0 = 1 - s \)
\[ G(s)H(s) = \frac{K(1-s)}{s(s^2 + 5s + 9)} \]

Characteristic equation is given by \( 1 + G(s)H(s) = 0 \)
\[ 1 + \frac{K(1-s)}{s(s^2 + 5s + 9)} = 0 \]
\[ s^2 + 5s + 9 = K \]
\[ s^2 + 5s + 9 - K = 0 \]
\[ s^1 = 9 - K \]
\[ s^0 = \frac{K}{5} \]

\[ \text{For stability:} \quad K > 0 \]
\[ \frac{K}{5} > 0 \quad \text{or} \quad K < 7.5 \]

\[ \therefore \text{Range of} \ K \quad 0 < K < 7.5 \] Ans.

Example 5.19. By means of Routh criterion, determine the stability of the systems represented by the following characteristic equations. For systems found to be unstable, determine the number of roots of the characteristic equation in the right half \( s \)-plane.

(a) \[ s^2 + 2s^2 + 4s^2 + 3 = 0 \]
(b) \[ s^2 + 2s^2 + 4s + 2 = 0 \]
(c) \[ s^3 + 3s^2 + 5s^2 + 8s^2 + 6s^2 + 4 = 0 \]
Solution: 

(a) \( s^2 + 2s + 8s + 4s + 3 = 0 \)

\[
\begin{array}{c}
s^2 \\
s^1 \\
s^0 \\
s^2 \\
s^1 \\
s^0 \\
\end{array}
\begin{array}{c}
1 \\
2 \\
4 \\
6 \\
3 \\
3 \\
\end{array}
\]

No. of sign change in first column = 0

No. of roots in right half of s-plane = 0

Hence system is stable.

(b) \( s^4 + 2s^3 + s^2 + 4s + 2 = 0 \)

\[
\begin{array}{c}
s^4 \\
s^3 \\
s^2 \\
s^1 \\
s^0 \\
\end{array}
\begin{array}{c}
1 \\
1 \\
2 \\
4 \\
2 \\
\end{array}
\]

No. of sign change in first column = 2

No. of roots in right half of s-plane = 2

Hence system is unstable.

(c) \( s^4 + 3s^3 + 5s^2 + 9s^2 + 8s^2 + 6s + 4 = 0 \)

\[
\begin{array}{c}
s^4 \\
s^3 \\
s^2 \\
s^1 \\
s^0 \\
\end{array}
\begin{array}{c}
1 \\
5 \\
8 \\
9 \\
6 \\
\end{array}
\]

The elements of fourth row are zero, hence form the auxiliary eqn

\[
A(s) = 2s^3 + 6s^2 + 4
\]

\[
\frac{dA(s)}{ds} = 8s^3 + 12s
\]

\[
\begin{array}{c}
s^3 \\
s^2 \\
s^1 \\
s^0 \\
\end{array}
\begin{array}{c}
1 \\
3 \\
2 \\
0 \\
\end{array}
\]

Since, no sign change in the first column, hence system is stable.

The roots of auxiliary eqn are

\[
\begin{align*}
s &= \pm j1 \\
s &= \pm j\sqrt{2}
\end{align*}
\]

Four roots lie on imaginary axis.

Example 5.20: Determine the range of values of \( K \) \((K > 0)\) such that the characteristic equation

\[
s^4 + 3(K + 1)s^2 + (7K + 5)s + (4K + 7) = 0
\]

has roots more negative than \( s = -1 \).

Solution: Put \( s = -1 \) in given characteristic eqn

\[
(z - 1)^4 + 3(K + 1)(z - 1)^2 + (7K + 5)(z - 1) + (4K + 7) = 0
\]

Example 5.21. The characteristic equation of feedback control system is

\[
s^4 + 20s^3 + 15s^2 + 2s + K = 0
\]

(a) Determine the range of \( K \) for the system to be stable.

(b) Can the system be marginally stable? If so, find the required value of \( K \) and the frequency of sustained oscillation.

Solution:

\[
\begin{align*}
s^3 &\quad 20 \\
s^2 &\quad 15 \\
s^1 &\quad K
\end{align*}
\]

For stability

\[
\begin{align*}
K &> 0 \\
\frac{3K + 2 - 4}{3K} &> 0 \\
3K^2 + 6K - 4 &> 0
\end{align*}
\]

Hence, the range is \( K > 0 \)

Example 5.22. The characteristic equation for a certain feedback control system is given below. Determine the range of value of \( K \) for the system to be stable.

\[
s^4 + 4s^2 + 13s^2 + 36s + K = 0
\]

Solution:

\[
\begin{align*}
s^3 &\quad 13 \\
s^2 &\quad 36 \\
s^1 &\quad K
\end{align*}
\]

Since, no sign change in the first column, hence system is stable.

The roots of auxiliary eqn are

\[
\begin{align*}
s &= \pm j1 \\
s &= \pm j\sqrt{2}
\end{align*}
\]

Four roots lie on imaginary axis.
5.7. ROOT LOCUS

Root locus is a graphical method in which roots of the characteristic equation are plotted in s-plane as the gain is varied from zero to infinity is called root locus.

Consider a unity feedback system as shown in fig. 5.10.

The characteristic equation is

\[ 1 + G(s) \cdot H(s) = 0 \]

\[ G(s) = \frac{K}{s(s+2)} \quad H(s) = 1 \]

\[ \frac{1 + K}{s(s+2)} = 0 \text{ or,} \]

\[ s^2 + 2s + K = 0 \]

The roots of eq (5.14) are

\[ s_1 = -1 + \sqrt{1-K} \quad s_2 = -1 - \sqrt{1-K} \]

As 'K' is varied, the two roots give the loci in s-plane. For various values of K, the location of the roots are:

1. When \( 0 < K < 1 \), the roots are real and distinct.
2. When \( K = 0 \), the two roots are \( s_1 = 0 \) and \( s_2 = -2 \). These are also the open loop poles.
3. When \( K = 1 \), both roots are real and equal.
4. When \( K > 1 \), the roots are complex conjugate with real part \(-1\).
5. When 'K' is varying the root locus is shown in the fig. 5.11.

(a) When \( K = 0 \), two branches of root locus starts from \( s = 0 \) & \( s = -2 \).
(b) When \( K = 1 \), both roots meet at \( s = -1 \).
(c) When \( K \to \infty \), the roots breakaway from the real axis and become complex conjugate having negative real part equal to \(-1\).

Consider a closed loop system shown in fig 5.12. The overall transfer function is

\[ \frac{G(s)}{R(s)} = \frac{G(s)H(s)}{1 + G(s)H(s)} \]

The characteristic equation is

\[ 1 + G(s) \cdot H(s) = 0 \]

\[ G(s) \cdot H(s) = -1 \]

\[ |G(s)H(s)| = -1 \]

\[ \angle G(s)H(s) = \pm 180 \text{ or } 2(K+1) \]

\[ K = 0, 1, 2, 3, \ldots \]

Equations (5.15) and (5.16) are the magnitude and angle conditions. The roots of the characteristic equations must satisfy the above two conditions or in other words the values of 's' which satisfies these conditions are the roots of the characteristic equation or poles of the closed loop.

Consider a system with \( G(s) \cdot H(s) = \frac{K}{s(s+4)(s+5)} \). Find whether \( s = -1 \) is on root locus or not using angle condition.

The angle condition is

\[ \angle G(s)H(s) = \pm 180^\circ (2K+1) \]

\[ \angle G(s)H(s)_{s=-1} = \frac{jK+j0}{(-1+j0)(3+j0)(4+j0)} \]

\[ = \frac{0^\circ}{180^\circ} = -180^\circ \]

Since \( \angle G(s)H(s) = -180^\circ \) at \( s = -1 \), this satisfies the angle condition, hence the point at \( s = -1 \) is on the root locus.

Consider above example for magnitude condition. Now we are interested to find the value of \( K \) at which \( s = -1 \) is one of the roots of \( 1 + G(s) \cdot H(s) = 0 \).

\[ |G(s)H(s)|_{s=-1} = 1 \]

\[ \frac{|K|}{|1-4+1|} = 1 \]

\[ K = 12 \]

Therefore for \( K = 12 \), one of the three roots is located at \( s = -1 \). Remaining two roots can also be easily calculated.

5.8. RULES FOR CONSTRUCTION OF ROOT LOCII

Following are the rules to sketch the root locus plot.

Rule 1. The root locus is symmetrical about the real axis.

Rule 2. The root loci starts from an open loop pole with \( K = 0 \) e.g. For the system having

\[ G(s) \cdot H(s) = \frac{K(s+3)}{(s+2)} \]

Find the starting point of the root loci.
Solution: According to the rule the root loci starts from $s = -2$.

Rule 3. The root loci will terminate either on an open loop zeros or on infinity with $K = \infty$. For the ending point of the root loci given in eqn. 5.17. According to the rule the root loci will terminate at $s = -3$.

Rule 4. If $N = \text{No. of separate loci}$

$P = \text{No. of finite poles}$

$Z = \text{No. of finite zeros}$

then the number of root loci will be equal to the no. of poles if number of poles are more than number of zeros i.e. $P > Z$

If $Z > P$, then number of root loci will be equal to the number of zeros.

If $P = Z$, then No. of root loci = Poles = Zeros.

E.g. Find the number of separate root loci for the system given by the eqn. 5.17.

$P = 1$

$Z = 1$

$N = 1$

Solution:

Rule 5. Root Loci on the Real Axis.

Any point on the real axis is a part of the root locus if and only if the number of poles and zeros to its right is odd.

Rule 6. Asymptotes

The branches of root locus tend to infinity along a set of straight line called asymptotes. These asymptotes making an angle with real axis and is given by

$$\phi = \frac{(2K+1)180^\circ}{P-Z}$$

where $K = 0, 1, 2, \ldots$

The total number of asymptotes = $P - Z$

E.g. If $G(s)H(s) = \frac{K}{s(s^2 + 6s + 10)}$

$P = 3$

$Z = 0$

No. of asymptotes = $P - Z = 3 - 0 = 3$

$K = 0$

$\phi_1 = \frac{(2 \times 0 + 1)180^\circ}{3} = 60^\circ$

$K = 1$

$\phi_2 = \frac{(2 \times 1 + 1)180^\circ}{3} = 180^\circ$

$K = 2$

$\phi_3 = \frac{(2 \times 2 + 1)180^\circ}{3} = 300^\circ$

Rule 7. Centroid of Asymptotes

The point of intersection of asymptotes with real axis is called centroid of asymptotes ($\sigma_c$) and is given by

$$\sigma_c = \frac{\text{sum of poles - sum of zeros}}{P-Z}$$

E.g. Find the centroid of asymptotes of the system given by eqn. 5.18.
Rule No. 10. The intersection of root locus branches with ju-axis can be determined through Hurwitz criterion.

e.g. If \( G(s)H(s) = \frac{K}{s(s^2 + 6s + 10)} \). Find the intersection of the root locus with the imaginary axis.

Solution: The characteristic \( \phi s^3 + 6s^2 + 10s + K = 0 \)

\[
\begin{align*}
    s^3 & : 1 \\
    s^2 & : 6 \\
    s^1 & : 60 - K \\
    s^0 & : K
\end{align*}
\]

Hence, we get a zero row if \( K = 60 \)

The auxiliary equation \( A(s) = 6s^2 + K \)

\[
\begin{align*}
    6s^2 + K & = 0 \\
    6s^2 + 60 & = 0 \\
    s & = \pm j3.16
\end{align*}
\]

The root locus branches cross the imaginary axis at \( s = \pm j3.16 \) for \( K = 60 \).

5.9. DETERMINATION OF \( K \) ON ROOT LOCI

\[ \text{Fig. 5.14.} \]

The value of \( K \) can be determined by

Product of all vector lengths drawn from the poles of \( G(s)H(s) \) to the point

\[ K = \frac{B.C}{A} \]

\( e.g. \) in Fig. 5.14 determine the value of \( K \) at the point of intersection of root locus branch with imaginary axis.

Example 5.23. The forward path transfer function of a unity feedback system is given by \( G(s)H(s) = \frac{K}{s(s+4)(s+5)} \). Sketch the root locus as \( K \) varies from zero to infinity.

Solution: Step 1: Plot the poles and zeros

Poles are at \( s = 0, -4 \) & \( -5 \)

No. of poles \( P = 3 \)
No. of zeros \( Z = 0 \)

Step 2: The root locus exists between \( s = 0 \) & \( s = -5 \) and to the left of -5. Mark the root locus on real axis.

Step 3: Number of root loci

\[ P = 3 \]

\[ Z = 0 \]

\[ N = 3 \]

Step 4: Centroid of asymptotes

\[ \sigma_A = \frac{\sum \text{poles} - \sum \text{zeros}}{P - Z} = \frac{(0 - 4 - 5) - (0)}{3 - 0} = -3 \]

Step 5: Angle of asymptotes

\[ \phi = \left( \frac{2K + 1}{P - Z} \right) \times 180^\circ \]

\[ K = 0, \quad \phi_1 = 60^\circ \]

\[ K = 1, \quad \phi_1 = 180^\circ \]

\[ K = 2, \quad \phi_1 = 300^\circ \]

Step 6: Calculation of breakaway point

The characteristic equation \( s^3 + G(s)H(s) = 0 \)

\[ \begin{align*}
    s^3 + 9s^2 + 20s + K & = 0 \\
    s^3 + 9s^2 + 20s & = -K \\
    \frac{ds}{dt} & = -3s^2 - 18s - 20 = 0 \\
    3s^2 + 18s + 20 & = 0 \\
    s_{1,2} & = -1.4725, \quad s_3 = -4.5275
\end{align*} \]

Since, -4 to -5 is not the segment of root locus. Therefore we consider -1.4725 as a breakaway point.

Step 7: Determination of point of intersection of branches of root locus with imaginary axis by Routh Hurwitz.

The characteristic \( \phi s^3 \) is

\[ \begin{align*}
    s^3 + 9s^2 + 20s + K & = 0 \\
    s^3 & : 1 \\
    s^2 & : 9 \\
    s^1 & : \frac{180 - K}{9} \\
    s^0 & : K
\end{align*} \]

For \( K = 180 \), the auxiliary \( \phi s^3 \) \( A(s) = 9s^2 + K \)

\[ \begin{align*}
    9s^2 + K & = 0 \\
    9s^2 + 180 & = 0 \\
    s & = \pm j 4.47
\end{align*} \]

The complete root locus is shown in Fig. 5.15.
Example 5.24. For a unity feedback system the open loop transfer function is given by

\[ G(s) = \frac{K}{s(s + 2)(s^2 + 6s + 25)} \]

(a) Sketch the root locus for \( 0 \leq K \leq \infty \).
(b) At what value of \( K \) the system becomes unstable.
(c) At this point of instability determine the frequency of oscillation of the system.

Solution:

Step 1: Plot the poles and zeros

Poles are at \( s_1 = 0, s_2 = -2, s_3 = -3 + j4, s_4 = -3 - j4 \)
\[ s^2 + 6s + 25 = 0 \]
\[ s_1 = -3 + j4 \]
\[ s_2 = -3 - j4 \]

Step 2: The segment on the real axis between \( s = 0 \) & \( s = -2 \) is the part of the root locus.

Step 3: Number of root loci

- Number of Poles \( P = 4 \)
- Number of zeros \( Z = 0 \)
- Number of root loci \( N = P = 4 \)

Step 4: Centroid of the asymptotes

\[ \sigma_s = \frac{\text{sum of poles} - \text{sum of zeros}}{P - Z} = \frac{0 - 2 - 3 + 4 - 3 - 4}{4 - 0} = -2 \]

Step 5: Angle of asymptotes

\[ \phi = \frac{2K + 1}{P - Z} 180^\circ \]
- \( K = 0 \):
  \[ \phi = \frac{2 \times 0 + 1}{4} 180^\circ = 45^\circ \]
- \( K = 1 \):
  \[ \phi = \frac{2 \times 1 + 1}{4} 180^\circ = 135^\circ \]
- \( K = 2 \):
  \[ \phi = \frac{2 \times 2 + 1}{4} 180^\circ = 225^\circ \]
- \( K = 3 \):
  \[ \phi = \frac{2 \times 3 + 1}{4} 180^\circ = 315^\circ \]

Step 6: Breakaway point

The characteristic eqn

\[ 1 + \frac{K}{s(s + 2)(s^2 + 6s + 25)} = 0 \]

or, \( K + s^2 + 8s^2 + 376s + 50s = 0 \)

\[ K = -(s^2 + 8s^2 + 376s + 50) \]

\[ \frac{dk}{ds} = -(4s^3 + 24s^2 + 74s + 50) = 0 \]

or, \( 4s^3 + 24s^2 + 74s + 50 = 0 \)

By trial & error method \( s = -0.8981 \)

* Put the values of \( s \) between 0 & -2 because the portion between 0 & -2 is the part of root locus, if there is no breakaway point then it will lie between these points.
Step 7: Determination of $\rho$ crossover (by Routh–Hurwitz)

\[ \begin{align*}
\text{ch. eq: } & s^4 + 8s^3 + 37s^2 + 50s + K = 0 \\
& s^4 \\
& s^3 \\
& s^2 \\
& s^1 \\
& K \\
& 137.5 - 8K \\
& 30.75 \\
& s \\
& 1537.5 - 8K \\
& 30.75 \\
& K \\
& 1537.5 - 8K = 0 \\
& K = 192.18
\end{align*} \]

For $K = 192.18$ Auxiliary $\rho$

\[ \begin{align*}
30.75 s^2 + K &= 0 \\
30.75 s^2 &= -192.18 \\
s &= \pm \frac{\sqrt{25}}{2}
\end{align*} \]

Step 8: Angle of departure from upper complex pole

\[ \begin{align*}
\phi_r &= 180° - (104° + 90° + 127°) \\
\phi_r &= -141°
\end{align*} \]

b. The range of values for stability is $0 < K < 192.18$

The closed loop system becomes unstable for $K < 0$ & $K > 192.18$

c. At this point of instability the gain is $K = 192.18$

\[ 30.75 s^2 + 192.18 = 0 \]

Put

\[ s = j\omega \]

\[ -30.75 \omega^2 + 192.18 = 0 \]

\[ \omega = 2.5 \text{ rad/sec.} \]

The frequency of oscillation at the point of instability $= 2.5 \text{ rad/sec.}$

The root locus plot is shown in fig 5.16.

Example 5.25. Consider a unity feedback control system with the following feedforward transfer function.

\[ G(s) = \frac{K}{s(s^2 + 4s + 8)} \]

Plot the root loci for the system.

Solution: Step 1: Plot the poles & zeros

\[ s^2 + 4s + 8 = 0 \]

\[ s_1, s_2 = \frac{-4 \pm \sqrt{16 - 32}}{2} = -2 \pm j2 \]

Three poles are at $s_1 = 0$, $s_2 = -2 + j2$ & $s_3 = -2 - j2$.

Step 2: Since there is only one pole at $s = 0$, the entire left half of the real axis is the part of the root locus.

Step 3: Number of root loci

\[ \begin{align*}
\text{No. of poles} &= P = 3 \\
\text{No. of zeros} &= Z = 0 \\
\text{No. of root loci} &= M = P = 3
\end{align*} \]
Step 4: Centroid of the asymptotes

\[ \sigma_c = \frac{\text{sum of poles} - \text{sum of zeros}}{P - Z} \]

\[ \sigma_c = \frac{0 - 2 + j2 + 2 - j2}{3} = -\frac{4}{3} = -1.33 \]

Step 5: Angle of asymptotes

\[ \phi = \frac{2K + 1}{P - Z} \cdot 180^\circ \]

\[ K = 0, \phi_1 = 60^\circ \]

\[ K = 1, \phi_2 = 180^\circ \]

\[ K = 2, \phi_3 = 300^\circ \]

Step 6: Breakaway point

The characteristic eqn \[ 1 + G(s) H(s) = 0 \]

\[ \frac{K}{s(s^2 + 4s + 8)} = 0 \]

\[ s^3 + 4s^2 + 8s + K = 0 \]

or,

\[ K = -(s^3 + 4s^2 + 8s) \]

\[ \frac{dK}{ds} = -(3s^2 + 8s + 8) \]

\[ 3s^2 + 8s + 8 = 0 \]

\[ s = \frac{-8 \pm \sqrt{64 - 96}}{6} = -1.33 \pm 0.943 \]

Since, at the point \( s = -1.33 \pm 0.943 \), the angle condition is not satisfied. Hence there is no breakaway point on the real axis.

Step 7: Point of intersection with imaginary axis.

The characteristic equation

\[ s^3 + 4s^2 + 8s + K = 0 \]

Routh array is

| \( s^3 \) | \( 1 \) | \( 8 \) |
| \( s^2 \) | \( 4 \) | \( K \) |
| \( s^1 \) | \( 32 - K \) | \( 4 \) |
| \( s^0 \) | \( K \) |

For sustained oscillation \( 32 - K = 0 \) or \( K = 32 \)

The auxiliary equation \( A(s) = 4s^2 + K \)

\[ 4s^2 + 32 = 0 \]

\[ s = \pm j\sqrt{8} \]

Step 8: The angle of departure from the upper complex pole is

\[ \phi_p = 180^\circ - (135^\circ + 90^\circ) = -45^\circ \]

The root locus is shown in fig. 5.17.
Example 5.26. Plot the root loci for the closed loop control system with

\[ G(s) = \frac{K}{s(s+1)(s^2 + 4s + 5)} \]

Solution: Step 1: Plot the poles and zeros \( s^2 + 4s + 5 \)

\[ s = \frac{-4 \pm \sqrt{16 - 20}}{2} = -2 \pm j 1 \]

Poles are at \( s_1 = 0; \ s_2 = -1 \), \( s_3 = -2 + j 1 \) & \( s_4 = -2 - j 1 \).

Step 2: The segment between \( s = 0 \) & \( s = -1 \) is the part of the root locus on real axis.

Step 3: Number of root loci.
Number of root loci \( N = P = 4 \).

Step 4: Centroid of the asymptotes

\[ \sigma_a = \frac{\text{sum of poles} - \text{sum of zeros}}{P - Z} = \frac{0 - 1 - 2 + 1 - 1 - 1 - 0}{4} = -5/4 = -1.25 \]

Step 5: Angle of asymptotes

\[ \phi = \frac{2K + 1}{P - Z} \times 180^\circ \]

\[ \begin{align*}
K = 0 & \quad \phi = 45^\circ \\
K = 1 & \quad \phi = 135^\circ \\
K = 2 & \quad \phi = 225^\circ \\
K = 3 & \quad \phi = 315^\circ \\
\end{align*} \]

Step 6: Breakaway point.

The characteristic equation

\[ 1 + G(s)H(s) = 0 \]

\[ \frac{K}{s(s+1)(s^2 + 4s + 5)} = 0 \]

\[ s^4 + 5s^3 + 9s^2 + 5s + K = 0 \]

\[ K = -(s^4 + 5s^3 + 9s^2 + 5s) \]

\[ \frac{dK}{ds} = -[4s^3 + 15s^2 + 18s + 5] \]

Breakaway point is when \( s = 0.4 \).

Step 7: Angle of departure at the upper complex pole

\[ \phi_j = 180^\circ \times (154^\circ - 156^\circ + 90^\circ) = -200^\circ \]

Step 8: Point of intersection on \( j\omega \) axis

\[ s^4 + 5s^3 + 9s^2 + 5s + K = 0 \]

\[ \begin{align*}
s^4 & = 1 \\
s^3 & = 9 \\
s^2 & = 5 \\
s & = \frac{40 - 5K}{K} \]

Step 5: Breakaway point.
characteristic equation

\[ 1 + \frac{K(s+1)}{s^2(s+3.6)} = 0 \]

\[ \frac{dK}{ds} = \frac{(s+1)(3s^2 + 7.2s) - (s^3 + 3.6s^2)}{(s+1)^2} \]

\[ s = 0 \text{ and } s = -3.3 \pm \frac{3.3^2 - 4 \times 1 \times 3.6}{2} \]

Point \( s = 0 \) is the actual breakaway point.

\( s = 0, s = -1.65 \pm j 0.936 \)

\( s = -1.65 \pm j 0.936 \) is a complex quantity, this will not be breakaway point.
Step 6: Point of intersection of root loci on \( j\omega \) axis.

The characteristic equation is:

\[ s^3 + 3.6s^2 + Ks + K = 0 \]

For sustained oscillation, \( K = 0 \)

\[ A(s) = 3.6 s^2 + K \]
\[ 3.6 s^2 + 0 = 0 \]
\[ s^2 = 0 \]

Root locus branches do not cross the \( j\omega \) axis.
The root locus is shown in Fig. 5.19.

Example 5.28. A unity feedback control system has an open loop transfer function

\[ G(s) = \frac{K}{s(s^2 + 4s + 13)} \]

Sketch the root locus plot of the system by determining the following:

(a) Centroid, number and angle of asymptotes.
(b) Angle of departure of root loci from the poles
(c) Breakaway point if any
(d) The value of \( K \) and the frequency at which the root loci cross \( j\omega \) axis.

Solution:

Step 1: Plot the poles & zeros

\[ s = 0 \]

\[ s = -\frac{4 \pm \sqrt{16 - 52}}{2} = -2 \pm j3 \]

Step 2: From \( s = 0 \) to the left is the part of the root locus.

Step 3: Number of root loci \( N = P = 3 \)

Step 4: Centroid of the asymptotes

\[ \sigma_A = \frac{\text{sum of poles} - \text{sum of zeros}}{P - Z} \]

\[ \sigma_A = \frac{0 - 2 + j3 - 2 - j3 - 0}{3 - 0} = \frac{-4}{3} = -1.33 \]

Step 5: Angle of asymptotes

\[ \phi = \frac{2K + 1}{P - Z} \times 180^\circ \]

\( K = 0 \)
\( \phi_1 = 60^\circ \)
\( K = 1 \)
\( \phi_2 = 180^\circ \)
\( K = 2 \)
\( \phi_3 = 300^\circ \)

Step 6: Breakaway point

The characteristic equation is

\[ 1 + \frac{K}{s(s^2 + 4s + 13)} = 0 \]
\[
K = -\{s^3 + 4s^2 + 13s\}
\]
\[
\frac{dK}{ds} = -\{3s^2 + 8s + 13\} = 0
\]
\[
s = \frac{-8 \pm \sqrt{64 - 156}}{6} = \frac{-8 \pm 9.59}{6} = -1.33 \pm 1.59
\]

Since, it is a complex root, there will be no breakaway point on real axis.

Step 7: Point of intersection of root locii on imaginary axis characteristic eqn:
\[
s^3 + 4s^2 + 13s + K = 0
\]
\[
s^2 = 1, 13
\]
\[
s = 4, K
\]
\[
s = \frac{52 - K}{4}
\]
\[
s = K
\]

For sustained oscillation \(52 - K = 0\) or \(K = 52\)

Auxiliary equation \(A(s) = 4s^2 + K\)
\[
4s^2 + 52 = 0
\]
\[
s^2 = -13
\]
\[
s = \pm j3.6
\]

\(\therefore\) The value of \(K\) at the point of intersection on imaginary axis = 52

The frequency at this point = 3.6 rad/sec.

Step 8: Angle of departure at the upper complex pole
\[
\phi_i = 180^0 - (124^0 + 90^0) = -34^0
\]

The root locus is shown in fig 5.19.

Example 5.29. Sketch the root locus for
\[
G(s)H(s) = \frac{K}{s(s + 2)(s + 4)}
\]

and evaluate the value of \(K\) at the point where the root locii crosses the imaginary axis. Also determine the frequency. Also, determine the value of \(K\) so that the dominant pair of complex poles of the system has a damping ratio of 0.5.

Solution: Step 1: Plot the poles and zeros.

Poles are at \(s_i = 0, s_2 = -2, s_3 = -4\)

Step 2: The segment between \(s = 0 \& -2\) and from \(s = -4\) to the left are the parts of root locus.

Step 3: Number of root locii \(N = P = 3\)

Step 4: Centroid of the asymptotes
\[
\sigma = \frac{\text{sum of poles} - \text{sum of zeros}}{P - Z}
\]
\[
\sigma = \frac{-2 - 4}{3 - 0} = -2
\]

Step 5: Angle of asymptotes
\[
\phi = \frac{2K + 1}{P - Z} \times 180^0
\]
Root locus of \( G(s) = \frac{K}{s^2 + 4s + 13} \)

---

\[ \begin{align*}
\theta_1 &= 60^\circ \\
\theta_2 &= 160^\circ \\
\theta_3 &= 300^\circ \\
\end{align*} \]

Step 6: Break-away point

The characteristic equation

\[ \frac{dK}{ds} = \frac{d}{ds}(s^2 + 8s + 16) = 0 \]

\[ 3s^2 + 12s + 8 = 0 \]

\[ s = -0.85 \text{ & } -3.15 \]

Since, -3.15 is not the part of root locus therefore break-away point is -0.85.

Step 7: Point of intersection of root loci on imaginary axis

Characteristic equation

\[ \sigma + 6\sigma + 8 + K = 0 \]

\[ \sigma = -1.8, 8 \]

\[ s = \frac{48 - K}{6} \]

For sustained oscillation, \( K = 48 \)

Auxiliary eqn

\[ A(s) = 6s^2 + K \]

\[ 6s^2 + 48 = 0 \]

\[ s = \pm \frac{1}{\sqrt{2}} \]

Put

\[ s = j\omega \]

Frequency of oscillation = \( 2.8 \text{ rad/sec} \)

The value of \( K \) at the point of intersection of root loci with the imaginary axis = 48

This value of \( K \) can also be determined from the graph. Draw the lines from all poles to the point of intersection on imaginary axis (c).

From graph

\[ \begin{align*}
AC &= 9.8 \text{ cm} = 4.9 \\
BC &= 6.9 \text{ cm} = 3.45 \\
OC &= 8.8 \text{ cm} = 2.9 \\
K &= 4.9 \times 3.45 \times 2.9 = 49
\end{align*} \]

In second quadrant draw a \( \zeta \)-line at \( \theta = \cos^{-1} 0.5 = 60^\circ \) with negative real axis. This line intersect the root loci at point D. The value of \( K' \) at this point can be obtained as:

Connect all the poles to this point and measure all the distance of all poles to this point

\[ \begin{align*}
OD &= 2.6 \text{ cm} = 1.3 \\
BD &= 3.6 \text{ cm} = 1.8 \\
AD &= 7.2 \text{ cm} = 3.6 \\
K &= 1.3 \times 1.8 \times 3.6 = 8.424
\end{align*} \]

The root locus is shown in the fig. 5.21.
Example 5.30. A unity feedback system has an open loop transfer function

$$G(s) = \frac{K(s+1)}{s(s-1)}$$

Sketch the root locus plot with 'K' as variable parameter and show that the loci of complex roots are part of a circle with (-1, 0) as centre and radius = $\sqrt{2}$.

Solution:
Step 1: Plot the poles & zeros
- Poles: $s_1 = 0, s_2 = 1$
- Zeros: $s_3 = -1$

Step 2: Centroid of asymptotes
$$\sigma_A = \frac{\text{sum of poles} - \text{sum of zeros}}{P - Z} = \frac{0 + 1 - (-1)}{2 - 1} = 2$$

Step 3: Angle of asymptotes
$$\phi = \frac{2K + 1}{P - Z} \times 180^\circ$$
$$K = 0,$$
$$\phi = \frac{2 \times 0 + 1}{2 - 1} \times 180^\circ = 180^\circ$$

Step 4: Breakaway point
The characteristic equation
$$1 + \frac{K(s+1)}{s(s-1)} = 0$$
$$K = -\frac{s(s-1)}{s+1}$$
$$\frac{dK}{ds} = s^2 + 2s - 1 = 0$$
$$s = -2.414 \text{ and } 0.414$$
Since, both values are the part of the root locus, hence there will be a breakaway point and other is re-entry or breaking point.

As $s = 0.414$ lies between two poles $s = 0$ and $s = 1$, therefore it is breakaway point and $s = -2.414$ lies between zero $s$: infinity therefore it is a breakin or re-entry point.

Step 5: Point of intersection on imaginary axis
characteristic equation
$$s^2(K - 1) s + 1 = 0$$
$$s^2 + 1 = 0$$
$$s = K - 1 > 0$$
$$K > 1$$
Auxiliary eq
$$A(s) = s^2 + 1$$
$$s = \pm j1$$
The root locus is shown in fig 5.22.
From the root locus it is clear that centre at (-1, 0) & radius = $\sqrt{2}$.
Example 5.3. In example 5.30 is the system stable for all values of $K$? If not determine the range of $K$ for stable system operation. Find also the marginal value of $K$ which causes sustained oscillations and the frequency of these oscillations.

Solution: In example 5.30 step 5 it is clear that the system is not stable for all values of $K$. The range of the $K$ for stable operation is

$$1 < K < \infty$$

Marginal value of $K$ for sustained oscillations $= 1$.

Frequency of sustained oscillations $= 1 \text{ rad/sec}$.

Example 5.32. The open loop transfer function of a control system is given by

$$G(s) H(s) = \frac{K}{s(s + 6)(s^2 + 4s + 13)}$$

Sketch the root locus and determine the breakaway point, the angle of departure from complex poles and the stability condition.

Solution: Step 1: Plot poles and zeros.

Poles are at

$$s_1 = 0, s_2 = -6$$

$$s^2 + 4s + 13 = 0$$

$$s = \frac{-4 \pm \sqrt{16 - 52}}{2} = -2 \pm j3$$

$$s_3 = -2 + j3, s_4 = -2 - j3$$

Step 2: The segment between $s = 0$ and $s = -6$ is the part of the root locus.

Step 3: Number of root loci $N = 4$

Step 4: Centroid of the asymptotes

$$\sigma_a = \frac{\text{sum of poles} - \text{sum of zeros}}{P - Z} = \frac{-6 - 2 + j3 - 2 - j3}{4} = 2.5$$

Step 5: Angle of asymptotes

$$\phi = \frac{2K + 1}{P - Z} \times 10^\circ$$

$$k^0$$

$$\phi_1 = 45^\circ$$

$$k^1$$

$$\phi_2 = 135^\circ$$

$$k^2$$

$$\phi_3 = 225^\circ$$

$$k^3$$

$$\phi_4 = 315^\circ$$

Step 6: Breakaway point

The characteristic equation $1 + G(s) H(s) = 0$

$$1 + \frac{K}{s(s + 6)(s^2 + 4s + 13)} = 0$$

or,

$$K = -(s^4 + 10s^3 + 37s^2 + 7s) = 0$$

$$\frac{dK}{ds} = -(4s^3 + 30s^2 + 74s + 78) = 0$$

$$4s^3 + 30s^2 + 74s + 78 = 0$$

By trial and error method $s = -4.2$

Breakaway point $= -4.2$
Step 7: Point of intersection of root loci on imaginary axis. The characteristic equation is
\[ s^4 + 10 s^3 + 37 s^2 + 78 s + K = 0 \]
Routh array
\[
\begin{array}{ccc}
 s^4 & 1 & 37 & K \\
 s^3 & 10 & 78 & \\
 s^2 & 29.2 & K & \\
 s^1 & 78 - 0.342 K & \\
 s^0 & K & \\
\end{array}
\]
For sustained oscillations, \( 78 - 0.34 K = 0 \) \( K = 229.41 \)
The auxiliary eq
\[ A(s) = 29.2 s^2 + K \]
\[ 29.2 s^2 + 229.41 = 0 \]
\[ s = \pm j 2.8 \]
Step 8: From Routh table. For stability
\( K > 0 \)
\[ 78 - 0.342 K > 0 \quad \text{or} \quad K < 229.41 \]
Step 9: The angle of departure from upper complex pole is
\[ \phi_u = 180^\circ - (124^\circ + 90^\circ + 37^\circ) \]
\[ \phi_u = -71^\circ \]
Similarly, the angle of departure from lower complex pole is
\[ \phi_d = +71^\circ \]
The root locus plot is shown in fig. 5.23.

Example 5.33. Draw the root locus for a system whose open loop transfer function is given by
\[ G(s)H(s) = \frac{K}{s(s + 4)(s^2 + 4s + 20)} \]
show all the salient points on the locus.

(Control system, R.M.I. University Faisalabad)

Solution: Step 1. Plot poles and zeros
Poles are:
\[ s = -2 \pm j4 \]
Step 2: The segment between \( s = 0 \) and \( s = -4 \) is the part of the root locus.
Step 3: Number of root loci \( N = P = 4 \)
Step 4: Centroid of asymptotes
\[ \sigma = \frac{\text{sum of poles} - \text{sum of zeros}}{P - Z} = \frac{0 - 4 - 2 + j4 - 2 - j4 + 0}{4} = -2 \]
Step 5: Angle of asymptotes
\[ \phi = \frac{2K + 1}{P - Z} \times 180^\circ \]
\[ K = 0 \quad \phi_0 = 45^\circ \]
\[ K = 1 \quad \phi_1 = 135^\circ \]
\[ K = 2 \quad \phi_2 = 225^\circ \]
\[ K = 3 \quad \phi_3 = 315^\circ \]
Step 6: Breakaway point

The characteristic equation \( 1 + G(s) H(s) = 0 \)

\[
\frac{1}{s(s+4)(s^2+4s+20)} = 0
\]

\[
s^4 + 8s^3 + 36s^2 + 80s + K = 0
\]

\[
K = -(s^4 + 8s^3 + 36s^2 + 80s)
\]

\[
\frac{dK}{ds} = -(4s^3 + 24s^2 + 72s + 80) = 0
\]

Breakaway point is \( s = -2 \) and two complex breakaway points are \( -2 \pm j2.45 \).

Step 7: Point of intersection of root loci on imaginary axis

Routh array

| \( s^4 \) | 1 | 36 | \( K \) |
| \( s^3 \) | 8 | 80 |
| \( s^2 \) | 26 | \( K \) |
| \( s^1 \) | 80 - 0.307K |
| \( s^0 \) | \( K \) |

For stability \( K > 0 \)

\[
80 - 0.307K > 0 \text{ or } K < 260
\]

at \( K = 260 \), the auxiliary eq

\[
A(s) = 26s^2 + K
\]

\[
26s^2 + 260 = 0
\]

\[
s = \pm j3.16
\]

Step 8: The angle of departure at upper complex pole

\[
\phi_c = 180^\circ - (117^\circ + 90^\circ + 63^\circ) = -90^\circ
\]

The root locus is shown in Fig. 5.24.

Example 5.34. Show that the root loci for a control system with

\[
G(s) = \frac{K(s^2 + 6s + 10)}{s^2 + 2s + 10}, H(s) = 1
\]

are the arcs of the circle centered at the origin with radius equal to \( \sqrt{10} \).

Solution:

\[
G(s) H(s) = \frac{K(s^2 + 6s + 10)}{s^2 + 2s + 10} = \frac{K(s + 3 - j1)(s + 3 + j1)}{(s + 1 - j3)(s + 1 + j3)}
\]

Put \( s = \sigma + j\omega \)

\[
G(s)H(s)_{s=\sigma+j\omega} = \frac{K[(\sigma + j\omega + 3 - j1)(\sigma + j\omega + 3 + j1)]}{[(\sigma + j\omega + 1 - j3)(\sigma + j\omega + 1 + j3)]}
\]

\[
= \frac{K[(\sigma + 3) + j(\omega - 1)][(\sigma + 3) + j(\omega + 1)]}{[(\sigma + 1) + j(\omega - 3)][(\sigma + 1) + j(\omega + 3)]}
\]

\[
\Rightarrow G(j\omega)H(j\omega) = \left[\tan^{-1} \frac{\omega - 1}{\sigma + 3} + \tan^{-1} \frac{\omega + 1}{\sigma + 3}\right] - \left[\tan^{-1} \frac{\omega - 3}{\sigma + 1} + \tan^{-1} \frac{\omega + 3}{\sigma + 1}\right]
\]

To satisfy the angle criterion, equate the above eq to \( 180^\circ \)

\[
\left[\tan^{-1} \frac{\omega - 1}{\sigma + 3} + \tan^{-1} \frac{\omega + 1}{\sigma + 3}\right] - \left[\tan^{-1} \frac{\omega - 3}{\sigma + 1} + \tan^{-1} \frac{\omega + 3}{\sigma + 1}\right] = 180
\]
Taking tangent of both sides
\[
\tan \left( \tan^{-1} \frac{\omega}{\sigma + 3} + \tan^{-1} \frac{\omega}{\sigma + 1} \right) - \tan \left( \tan^{-1} \frac{\omega}{\sigma + 3} + \tan^{-1} \frac{\omega}{\sigma + 1} \right) = \tan 180^\circ
\]

or,
\[
\frac{-\omega \cdot (\omega - 1)}{(\sigma + 3)^2} + \frac{\omega}{\sigma + 3} + \frac{\omega}{\sigma + 1} = \frac{-\omega \cdot (\omega - 1)}{(\sigma + 3)^2} + \frac{\omega}{\sigma + 1} + \frac{\omega}{\sigma + 3}
\]

or,
\[
\frac{2\omega}{(\sigma + 3)^2} = \frac{2\omega}{(\sigma + 1)^2}
\]

or,
\[
\frac{(\sigma + 3)^2 - (\omega^2 - 1)}{(\sigma + 1)^2} = \frac{(\sigma + 1)^2 - (\omega^2 - 9)}{(\sigma + 1)^2}
\]

or,
\[
(\sigma + 1) \left((\sigma + 3)^2 - (\omega^2 - 1)\right) = (\sigma + 3) \left((\sigma + 1)^2 - (\omega^2 - 9)\right)
\]

or,
\[
2\sigma^2 + 2\omega^2 - 20 = 0
\]

or,
\[
\sigma^2 + \omega^2 = 10
\]

or,
\[
(\sigma - 0)^2 + (\omega - 0)^2 = \left(\sqrt{10}\right)^2
\]

This is the equation of the circle with centre at origin and radius $= \sqrt{10}$ Proved.

Example 5.35. Show that the root loci for a control system with
\[
G(s) = \frac{K(s + 1)}{s(s - 1)} \quad H(s) = 1
\]
is the circle with centre at $(-1, 0)$ and radius $\sqrt{2}$

Solution:
\[
G(s)H(s) = \frac{K(s + 1)}{s(s - 1)}
\]

Put $s = \sigma + j\omega$

\[
\lim_{s \to \sigma + j\omega} G(s)H(s) = \frac{K(\sigma + j\omega + 1)}{(\sigma + j\omega)(\sigma + j\omega - 1)}
\]

\[
\frac{\partial G(s)H(s)}{\partial \sigma}_{s=\sigma+j\omega} = \sigma + j\omega + 1 - \sigma + j\omega - \sigma + j\omega - 1 = 0
\]

To satisfy the angle criterion
\[
\tan^{-1} \frac{\omega}{\sigma + 1} - \tan^{-1} \frac{\omega}{\sigma + 3} = 180^\circ
\]

Taking tangent on both sides
\[
\tan^{-1} \frac{\omega}{\sigma + 1} - \tan^{-1} \frac{\omega}{\sigma + 3} = 180^\circ
\]

\[
\frac{\omega}{\sigma + 1} - \frac{\omega}{\sigma + 3} = \frac{\omega}{\sigma + 1} + \frac{\omega}{\sigma + 3}
\]

or,
\[
\sigma + 3 - \omega^2 \sigma + 3 = \omega + 1 \sigma + 1
\]

or,
\[
2\sigma^2 + 2\omega^2 - 20 = 0
\]

or,
\[
\sigma^2 + \omega^2 = 10
\]

or,
\[
(\sigma - 0)^2 + (\omega - 0)^2 = \left(\sqrt{10}\right)^2
\]

This is the equation of the circle with centre at $(-1, 0)$ and radius $\sqrt{2}$ (Proved).

Example 5.36. The open loop transfer function of a system is given by
\[
G(s)H(s) = \frac{K(s + 12)}{s^2(s + 20)}
\]

Sketch the root locus for the system.

Solution: Step 1: Plot the poles and zero

Poles are at $s = 0$, $s = 0$, $s = -20$

zero is at $s = -12$

Step 2: The segment between $s = -20$ and $s = -12$ is the part of the root locus.

Step 3: Centroid of asymptotes
\[
\sigma_c = \frac{\text{sum of poles} - \text{sum of zeros}}{p - z} = \frac{0 + 0 - 20 + 12}{3 - 1} = -4
\]

Step 4: Angle of asymptotes
\[
\phi = \frac{2K + 1}{p - z} 180^\circ
\]

$k = 0 \quad \phi = 90^\circ$

$k = 1 \quad \phi = 270^\circ$

Step 5: Breakaway point! The characteristic eqn
\[
1 + \frac{K(s + 12)}{s^2(s + 20)} = 0
\]

or,
\[
K = \frac{(s^3 + 20s^2)}{s + 12}
\]

\[
\frac{dK}{ds} = \left[\frac{(s + 12)(3s^2 + 40s) - (s^3 + 20s^2)}{(s + 12)^2}\right] = 0
\]
or, \[ s^3 + 28s^2 + 240s = 0 \]
\[ s(s^2 + 28s + 240) = 0 \]

we get \[ s = 0, -14 \pm j6.63 \]

Breakaway point \( s = 0 \), points \( -14 \pm j6.63 \) are neither breakaway point nor breakin point, because the corresponding gain values \( K \) becomes complex quantities.

**Step 6:** Point of intersection of rod loci with imaginary axis.

The characteristic eqn
\[ s^3 + 20s^2 + ks + 12K = 0 \]

Put \( s = j\omega \)

\[
(j\omega)^3 + 20(j\omega)^2 + K(j\omega) + 12K = 0
\]

\[
(12K - 20\omega^2) + j\omega (k - \omega^2) = 0
\]

if \( \omega = 0, K = 0 \)

because of double pole at the origin, the root locus is tangent to the imaginary axis at \( s = j\omega' \).

The root locus is shown in fig. 5.25.

**Example 5.37.** Sketch the root locus for

\[ G(s) = \frac{K}{s(s^2 + 6s + 12)} \cdot H(s) = 1 \]

**Solution:**

**Step 1:** Draw the pole zero plot

poles at \( s = 0 \)

\[ s^2 + 6s + 12 = 0 \]

\[ s = -3 \pm j1.73 \]

**Step 2:** The left portion of the negative real axis from origin is the part of the root locus.

**Step 3:** Centroid of asymptotes

\[ \sigma_s = \frac{\text{sum of poles} - \text{sum of zeros}}{P-Z} = \frac{0 - 3 + j1.73 - 3 - j1.73}{3} = -2 \]

**Step 4:** Angle of asymptotes

\[ \phi = \frac{2K + 1}{P-Z} 180^\circ \]

\( K = 0 \)

\( \phi = 60^\circ \)

\( K = 1 \)

\( \phi = 180^\circ \)

\( K = 2 \)

\( \phi = 300^\circ \)

**Step 5:** Breakaway point

The characteristic equation

\[ 1 + \frac{K}{s(s^2 + 6s + 12)} = 0 \]

\[ K = -(s^2 + 6s + 12) \]

\[ \frac{dK}{ds} = -(3s^2 + 12s + 12) = 0 \]

\[ 3s^2 + 12s + 12 = 0 \]

\[ s = -2 \] is the breakaway point.
Step 6: Point of intersection of root loci on imaginary axis

Routh array

\[
\begin{array}{ccc}
\sigma^3 & 12 \\
\sigma^2 & 6 \\
\sigma^1 & \frac{72-K}{6} \\
\sigma^0 & K \\
\end{array}
\]

for stability \( K > 0 \)

\[
\frac{72-K}{6} > 0 \quad \text{or} \quad K < 72
\]

For \( K = 72 \)

the auxiliary equation \( A(s) = 6s^2 + K \)

\[
6s^2 + 72 = 0
\]

\[
s = \pm 3.46
\]

Step 7: Angle of departure from upper complex pole

\[
\phi_u = 180^\circ - (150^\circ + 90^\circ)
\]

\[
\phi_u = -60^\circ
\]

similarly from lower complex pole

\[
\phi_l = +60^\circ
\]

The root locus is shown in fig. 5.26.

Example 5.38. The block diagram of a position control system is shown in fig. 5.27. Draw the root locus for the system as the parameter \( \alpha \) varies from zero to infinity and hence determine the value of \( \alpha \) so that the damping ratio of the dominant poles is 0.6.

\[
\begin{array}{c}
R(s) \\
\quad \overset{\times}{\rightarrow} \\
10 + \frac{\alpha}{s} \\
\quad \leftarrow \frac{1}{s(s+6)} \\
C(s)
\end{array}
\]

Fig. 5.27.

Solution:

\[
G(s) = \frac{10s + \alpha}{s^2(s+6)}, \quad H(s) = 1.
\]

\[
C(s) = \frac{G(s)}{1 + G(s)H(s)}
\]

\[
R(s) = \frac{10s + \alpha}{s^2(s+6) + 10s + \alpha}
\]

The characteristic equation \( s^2(s+6) + 10s + \alpha = 0 \)

or, \( 1 + \frac{\alpha}{s^2(s+6) + 10s} = 0 \)

let \( G'(s) = \frac{\alpha}{s^2(s+6) + 10s} = \frac{\alpha}{s^2 + 6s + 10} \)

Step 1: Poles are at \( s_1 = 0, \ s_2 = -3 + j1, \ s_3 = -3 - j1 \)

Step 2: The entire left half of real axis is the part of root locus.
Step 3: The centroid of the asymptotes
\[ \sigma_a = \frac{\text{sum of poles - sum of zeros}}{P - Z} = \frac{0 - 3 + j1 - 3 - j1}{3} = -2 \]

Step 4: Angle of asymptotes
\[ \phi = \frac{2K + 1}{P - Z} \times 180^\circ \]
- \( K = 0 \) \( \phi_1 = 60^\circ \)
- \( K = 1 \) \( \phi_2 = 180^\circ \)
- \( K = 2 \) \( \phi_3 = 300^\circ \)

Step 5: The characteristic equation
\[ \alpha = -(s^3 + 6s^2 + 10s) \]
\[ \frac{d\alpha}{ds} = -3s^2 - 12s - 10 \]
\[ s = -1.18, -2.816 \]
\[ \frac{d^2\alpha}{ds^2} = -6s - 12 \]
Put \( s = -1.18 \)
\[ \frac{d^2\alpha}{ds^2} = +7.08 - 12 = -4.92 \]
Since, \( \frac{d\alpha}{ds} \) is negative for \( s = -1.18 \)

\[ \therefore \text{Breakaway point} = -1.18 \]
Put \( s = -2.816 \)
\[ \frac{d^2\alpha}{ds^2} = +4.896 \]
Since \( \frac{d^2\alpha}{ds^2} \) is positive for \( s = -2.816 \), therefore \( s = -2.816 \) is the reentry point.

Step 6: Point of intersection of root loci on \( j\omega \) axis
\[
\begin{array}{c|c|c}
\omega^3 + 6\omega^2 + 10\omega + \alpha &= 0 \\
\omega^3 &= 1 \\
\omega^2 &= 6 \\
\omega &= 60 - K \\
\omega^2 &= \alpha \\
\end{array}
\]
At \( K = 60 \), auxiliary equation \( A(\omega) = 6\omega^3 + \alpha \)
\[ 6\omega^3 + 60 = 0 \]
\[ \therefore \omega = \pm \sqrt[3]{3.16} \]
Step 7: Angle of departure from upper complex pole.
\[ \theta_v = 162^\circ \]
\[ \theta_v = 90^\circ \]
\[ \phi = 190^\circ -(\theta_v + \theta_v) = 180^\circ - (162^\circ + 90^\circ) = -72^\circ \]
The root locus plot is shown in the fig 5.28.
Step 8: Draw $\zeta$-line at $\cos^{-1} 0.6 = 53.1^\circ$ from negative real axis. This line intersects the root locus at point A. Measure the distance of point B and point C from point A.

\[ OA = 1.55, \ AB = 2.1, \ AC = 3.05 \]

Required value of $\alpha = 1.55 \times 2.1 \times 3.05 = 9.927$

Example 5.39. For the example 5.29 determine the phase margin and gain margin for $K = 12$.

Solution. The root locus is shown in Fig. 5.29.

GAIN MARGIN. (a) The desired value of $K = 12$

For marginal stability the value of $K = 48$ (from example 5.29)

\[
\text{Gain margin} = \frac{\text{Value of } K \text{ for marginal stability}}{\text{Desired value of } K} = \frac{48}{12} = 4
\]

Gain margin in $\text{db} = 20 \log_{10} 4 = 12.04 \text{ db}$

(b) For the root at $K = 12$, locate the point 'x' by trial and error method (the value of $K$ at this point should be 12). In this case

\[ Ox = 1.55, \ Ax = 3.8, \ Bx = 2.05 \]

At the point $x$: \[
\frac{K}{0x||Ax||Bx} = 1
\]

\[
\frac{K}{1.55 \times 3.8 \times 2.05} = 1 \quad \therefore \quad K = 12.07
\]

The roots at this point = $-0.5 + j 1.5$

(c) For P.M at $K = 12$, locate a point on imaginary axis such that $|G(j\omega) \ H(j\omega)| = 1$ (by trial and error method)

In present case it is 1.2 rad/sec.

\[
\frac{12}{\sqrt{2.1^2} + (1.2)^2} = 1.02
\]

Hence $\omega_n = 1.2 \text{ (rad/sec.)}$, is the gain crossover frequency

\[
P.M. = 180^\circ - (\theta_1 + \theta_2 + \theta_3) = 180^\circ - (90^\circ + 32^\circ + 17^\circ) = 41^\circ
\]

Example 5.40. For the system shown in fig. 5.30 determine the value of $\alpha$ so that the poles of the closed loop system transfer function have a damping ratio of 0.6.

Solution: For the above fig.

\[ C(s) = \frac{K}{s(s + 2)} \]

Characteristic equation:

\[ s^2 + 2s + 25s + 25 = 0 \]

or,

\[ s^2 + 2s + 25 + 25\alpha s = 0 \]
or, \[ \frac{25 \alpha e}{s^2 + 2s + 25} = 0 \]

Put \( K = 25 \alpha \)

\[ 1 + \frac{Ks}{s^2 + 2s + 25} = 0 \]

The given system has a zero at \( s = 0 \) and two poles at \( -1 \pm j4.89 \), there is a possibility of circle root locus. Since, the angle condition is

\[ \frac{Ks}{s^2 + 2s + 25} = \pm 180^\circ(2K + 1). \]

We have \( \pm \frac{\omega + 4.89}{\sigma + 1 + j4.89} - \frac{\omega - 4.89}{\sigma + 1 - j4.89} = \pm 180^\circ(2K + 1) \)

Put \( s = \sigma + j\omega \)

\[ \frac{\sigma + 1 + j4.89 + \omega}{\sigma + 1} + \frac{\sigma + 1 + j(\omega - 4.89)}{\sigma + 1} = \frac{\sigma + j\omega + 180^\circ}{\sigma} \]

\[ \tan^{-1} \frac{\omega + 4.89}{\sigma + 1} + \tan^{-1} \frac{\omega - 4.89}{\sigma + 1} = \tan^{-1} \frac{\omega}{\sigma} \]

Taking tangent on both sides we get

\[ \frac{\omega + 4.89}{\sigma + 1} + \frac{\omega - 4.89}{\sigma + 1} = \frac{\omega}{\sigma}\]

\[ 1 - \frac{\omega^2}{\sigma + 1} = \frac{\omega}{\sigma} \]

or, \[ 2\sigma (\sigma + 1) = (\sigma + 1)^2 - \omega^2 + 23.9121 \]

or, \[ \sigma^2 + \omega^2 = 24.9121 \]

or, \[ \sigma^2 + \omega^2 = 25 \]

This is the equation of the circle\(^*\) with center at origin and radius is 5.

We require \( \zeta = 0.6 \), draw a line at an angle \( \cos^{-1}(0.6) = 53.13^\circ \) with negative real axis. This line intersects the root locus at \( 'x' \). The value of \( K \) at \( 'x' \) is

\[ K = \frac{D_1 D_3}{D_2} = \frac{2.2 \times 9.1}{5} = 4.004 \]

Since,

\[ 25\alpha = K \]

\[ 25\alpha = 4.00 \alpha = 0.16016 \]

The root locus is shown in fig. 5.31.

The root at the point of intersection is \(-3 + j3.95\).

**5.10. EFFECT OF ADDITION OF POLES AND ZEROS ON ROOT LOCUS**

Consider \( G(s) H(s) = \frac{K}{s(s+4)} \)

\* For \( K > 0 \), the portion of the circle lies to the left of the complex poles. For \( K < 0 \) the portion of the circle lies to the right of the complex poles. Hence, for present case, it is not the root locus, where \( K > 0 \).
The root locus is shown in fig. 5.32(a).

Now add one pole at \( s = -5 \), then

\[
G(s)H(s) = \frac{K}{s(s+4)(s+5)}
\]

the corresponding root locus is shown in fig. 5.15.

So, before addition of pole for any value of \( K \) the system is stable but after addition of pole is the left half, the two branches of root locus moves to the right half for some value of \( K \). Therefore the system will be stable for this value of \( K \) after this value of \( K \) the system becomes unstable. So, the stability of the system gets restricted. For further addition of poles to left half, the breaking point moves towards right. Therefore we can say that by addition of poles to left half, the root locus shifts towards right hand side and stability of the system decreases.

Now consider open loop transfer function

\[
G(s)H(s) = \frac{K}{s(s+2)}
\]

The root locus is shown in fig. 5.32(b).

Now add zero to left half at \( s = -3 \), the root locus is shown in fig. 5.32(c).

\[
G(s)H(s) = \frac{K(s+3)}{s(s+2)}
\]

It can be seen that by addition of zero towards left, the root locus shifts towards left half. Since root locus shifts towards left half, the relative stability increases.

In conclusion we can say that by addition of poles, the root locus shifts towards imaginary axis and system stability decreases, while by addition of zeros towards left half, the root locus moves away from the imaginary axis & system stability increases.

5.11. INTRODUCTION

Consider the closed loop transfer function

\[
\frac{C(s)}{R(s)} = T(s) = \frac{G(s)}{1 + G(s)H(s)}
\]

The characteristic equation can be obtained by equating the denominator polynomial to zero. The characteristic equation is \( 1 + G(s)H(s) = 0 \)

\( G(s)H(s) \) is the loop transfer function, \( L(s) \)

when \( G(s)H(s) \) is the closed loop transfer function, \( L(s) \)

Closed loop transfer function poles : zeros of \( 1 + G(s)H(s) \)

Zeros of \( 1 + G(s)H(s) \) : Roots of the characteristic equation.

For the stability of the system we have already studied the Routh Hurwitz and root locus technique. For stability of the system, the roots of the characteristic equation should not lie in the right half of the \( s \)-plane or on the imaginary axis. In other words, we can say that a system is said to be closed loop stable if the poles of the close loop transfer function or zeros of \( 1 + G(s)H(s) \) are all in the left half of \( s \)-plane. Similarly for open loop stability, a system is said to be open loop stable if the poles of the loop transfer function are all in the left half of \( s \)-plane.

Now we are going to study the Nyquist stability criterion. The Nyquist stability criterion is based on the principle of argument. The principle of argument is related with the theory of mapping. So let we will study the mapping technique and principle of argument then Nyquist stability criterion.

5.12. MAPPING

Consider a function \( D(s) = s^2 + 1 \)

Any point in the \( s \)-plane can be mapped by locating the values of \( u \) and \( v \) for the given value of \( s \).

\[
D(s) = D(2 + j4) = (2 + j4)^2 + 1 = -11 + j16
\]

The mapping is shown in the fig. 5.33.

Therefore, we can say that every point in the \( s \)-plane maps into one and only one point in the \( D(s) \)-plane. Any close contour in the \( s \)-plane maps into the closed contour in the \( D(s) \)-plane. These are the properties of the mapping.

5.13. MAPPING OF CLOSE CONTOUR AND PRINCIPLE OF ARGUMENT

Consider the characteristic equation

\[
D(s) = 1 + G(s)H(s) = \frac{K(s-a_1)}{s-b_1(s-a_2)}...
\]

where \( a_1, a_2, ... \) are zeros and \( b_1, b_2, ... \) are poles. Let 's' be a complex variable \( s = 6 + j6 \), then, \( D(s) \) will also be complex

\[
D(s) = U(s) + jV(s)
\]
Now choose a closed path 'C' arbitrarily. The contour 'C' in s-plane is mapped onto the D(s) plane as contour \( \Gamma \).

![Figure 5.34](image)

Now, consider a contour 'C' in s-plane. This contour encircles neither zeros nor poles, then the contour 'T' in D(s) plane cannot encircle the origin or the point at infinity. The inside region of the contour 'C' does not have any pole or zero. If we trace the contour 'C' in clockwise direction then the inside region will be on right of the contour. The corresponding region in D(s) plane which contains neither origin nor the point at infinity should also be to the right of the contour \( \Gamma \).

If the contour 'C' encircles a zero but not poles then the point \( D_z(s) \) is given by

\[
[D_z(s)] = \frac{s-a_1}{s-b_1} \frac{s-a_2}{s-b_2} \ldots
\]

i.e.,

\[
D_z(s) = r^\phi
\]

It means that the tip of \( D_z(s) \) forms a close contour about the origin in clockwise direction. Similarly, if the contour 'C' encircles 'z' zeros in clockwise direction then the contour in D(s) plane encircles the origin of D(s) plane 'z' times in clockwise direction. In fig. 5.36(a) the contour 'C' encircles two zeros, the corresponding contour in D(s) plane is shown in fig. 5.36(b).

![Figure 5.35](image)

![Figure 5.36](image)

If any point traces the curve 'C' in clockwise direction and return to the starting point then that phase will generate an angle -2\( \pi \).

Consider another case when a pole is encircled by the contour 'C' in clockwise direction then the angle will be -2\( \pi \). The phasers \( -2\pi \) will also generate an angle -2\( \pi \) but due to in denominator the angle will be +2\( \pi \). Therefore it means that if the contour 'C' encircles 'P' number of poles in clockwise direction then the corresponding contour \( \Gamma \) will encircle the origin \( P \) times in counter clockwise direction.

![Figure 5.37](image)

Now, if contour 'C' encircles both zeros and poles in clockwise direction then the corresponding contour \( \Gamma \) encircles the origin of D(s) plane \( Z-P = N \) times in clockwise direction.

This relation between enclosures of zeros and poles by the contour in s-plane and encirclement of origin by the contour 'T' in D(s) plane is known as principle of argument.

**Conclusion:** \( N = Z - P \)

(a) If \( N > 0 \), i.e., \( Z > P \), then \( N \) is positive integer. In this case \( \Gamma \) will encircle the origin \( N \) times in the same direction as that of contour 'C'.

(b) If \( N = 0 \), i.e., \( Z = P \), the contour 'C' will not encircle the origin.

(c) If \( N < 0 \), i.e., \( Z < P \), then \( N \) is negative integer, in this case \( \Gamma \) will encircle the origin \( N \) times in opposite direction as that of 'C'.

For example, the contour 'C' encircles three poles and one zero in clockwise then the number of encirclement \( N = 1 - 3 = -2 \). Then the contour 'T' will encircle the origin two times in counterclockwise direction.

**5.14 NYQUIST PATH OR NYQUIST CONTOUR**

The overall transfer of a system is given by

\[
\frac{C(s)}{R(s)} = \frac{G(s)}{1 + G(s)H(s)}
\]

The characteristc equation is \( 1 + G(s)H(s) = 0 \).
The main purpose in study the stability of the closed loop system is to determine whether the characteristic equation has any root in the right half of s-plane i.e whether $C(s)/R(s)$ has any poles in the right half of s-plane.

For this purpose we use a contour in s-plane which encloses the entire right half plane. The contour having the encirclement in clockwise direction and radius 'R' approaches infinity. This path or contour is known as Nyquist contour (shown in fig. 5.38a). If the system does not have any poles or zero at origin then the contour is shown in fig. 5.38(b).

5.15. NYQUIST CRITERION

The characteristic equation is given by

$$D(s) = 1 + G(s)H(s)$$

The zeros of $D(s)$ are the roots of the characteristic equation. For a feedback system the necessary and sufficient condition is that all zeros of $1 + G(s)H(s)$ that is the roots of the characteristic equation must have negative real part i.e., they must lie in the left half of s-plane. In order to determine the presence of zeros in right half of s-plane we choose a contour as shown in fig 5.38 called Nyquist contour. Let there are 'Z' zeros and 'P' poles in the right half of s-plane. If this contour is mapped on to the D(s)-plane as $\Gamma_2$ then $\Gamma_2$ encloses the origin Z times (where $N = Z - P$) in clockwise. Hence the system is unstable because the clockwise encirclement is possible only when there are zeros of $D(s)$ in right half of s-plane.

A feedback system (closed loop system) is stable if and only if there is no zeros of $D(s)$ in the right half of s-plane i.e., $Z = 0$.

Therefore, for a closed loop system to be stable, the number of counter-clockwise encirclement of the origin of $D(s)$ plane by $\Gamma_2$ should equal the number of right half s-plane poles of $D(s)$ which are the poles of open loop transfer function $G(s)H(s)$.

Since

$$D(s) = 1 + G(s)H(s)$$

or,

$$G(s)H(s) = D(s) - 1$$

The contour $\Gamma_2$ in $D(s)$ plane can be mapped in $G(s)$ $H(s)$ plane. $\Gamma_2$ by shifting horizontally to the left by one unit. Thus the encirclement of the origin by the contour $\Gamma_2$ is equivalent to the encirclement of the point $(-1+j0)$ by the contour $\Gamma_2$ as shown in fig. 5.39.

In most single loop feedback system $G(s)H(s)$ has no poles in the right half plane i.e., $P = 0$ the closed loop system is stable if $N = P = 0$.

So, we can say that a closed loop system with $P = 0$ is stable if the net encirclement of the origin of $D(s)$ plane by $\Gamma_2$ contour is zero.

Now we can state the Nyquist stability criterion as follows:

A feedback system or closed loop system is stable if the contour $\Gamma_2$ of the open loop transfer function $G(s)H(s)$ corresponding to the Nyquist contour in s-plane encircles the point $(-1+j0)$ to counterclockwise direction and the number of counterclockwise encirclement about the point $(-1+j0)$ equals the number of poles of $G(s)H(s)$ in the right half of s-plane i.e., with positive real parts.

In common case of open loop stable system, the closed loop system is stable if the contour $\Gamma_2$ of $G(s)H(s)$ does not pass through or does not encircle $(1+j0)$ point, i.e net encirclement is zero.

5.16. GENERAL CONSTRUCTION RULES OF THE NYQUIST PATH

Consider the fig. 5.37

<table>
<thead>
<tr>
<th>Table 5.1</th>
</tr>
</thead>
<tbody>
<tr>
<td>Path ab</td>
</tr>
<tr>
<td>Path hc</td>
</tr>
<tr>
<td>Path cd</td>
</tr>
<tr>
<td>Path def</td>
</tr>
<tr>
<td>Path gh</td>
</tr>
<tr>
<td>Path hi</td>
</tr>
<tr>
<td>Path ija</td>
</tr>
</tbody>
</table>

Step 1: Check $G(s)$ for poles on $j\omega$ axis at and at the origin
Step 2: Using fig 5.25 to 5.27 sketch the image of the path $s - d$ in the $G(s)$-plane. If there are no poles on $j\omega$ axis equation (5.26) need not be employed.
Step 3: Draw the mirror image about the real axis of the sketch resulting from step 2.
Step 4: Use eq (5.28) plot the image of path def. This path at infinity usually plot into a point in the $G(s)$-plane.
Step 5: Use eq (5.32) plot the image of path ija (pole at origin).
Step 6: Connect all curves drawn into the previous steps.

Example 5.1: Determine the closed loop stability of a control system whose open loop transfer function is

$$G(s)H(s) = \frac{K}{s(1+sT)}$$

Solution: Given that

$$G(s)H(s) = \frac{K}{s(1+sT)}$$

Putting $s = j\omega$

$$G(j\omega)H(j\omega) = \frac{K}{j\omega(1+j\omega T)}$$

Rationalizing the equation (5.33) and separating into real and imaginary parts.

$$G(j\omega)H(j\omega) = \frac{K}{1+\omega^2 T^2} - j\frac{KT}{1+\omega^2 T^2}$$

$$\lim_{\omega \to 0} |G(j\omega)H(j\omega)| = \infty$$

$$\lim_{\omega \to 0} \angle G(j\omega)H(j\omega) = -90^\circ$$

$$\lim_{\omega \to \infty} |G(j\omega)H(j\omega)| = 0$$

$$\lim_{\omega \to \infty} \angle G(j\omega)H(j\omega) = -180^\circ$$

(type 'T' system)
Rationalize the equation (5.35) and separate the real and imaginary parts.

\[ \frac{1}{(1 + \omega^2 T_1^2)(1 + \omega^2 T_2^2)} = \frac{1}{(1 + \omega^2 T_1^2)(1 + \omega^2 T_2^2)} \]

Equate the real part to zero, we get

\[ \omega = \frac{1}{\sqrt{T_1 T_2}} \]

\[ G(j\omega) \mid_{\omega = \frac{1}{\sqrt{T_1 T_2}}} = \frac{K}{T_1 T_2} \]

---

The polar plot will lie in third quadrant.

The Nyquist plot is shown in Fig 5.40. The part for \( 0 < \omega < \infty \) is drawn (1) (2) and for \( \omega < 0 \) is shown by the point (3) (4) which is the mirror image of (1) (2). The semicircular detour around the origin in s-plane is mapped into a semicircular path of infinite radius representing a change of phase from + \( \pi \) to - \( \pi \).

As the point \((-1 + jo)\) is not encircled by the plot, \( N = 0 \)

\[ N = 0 \quad P = 0 \quad Z = 0 \]

The number of zeros or roots of the characteristic equation with positive real part is nil as hence the closed loop system is stable.

Example 5.42. Sketch the Nyquist plot and determine the stability of a unity feedback control system.

\[ G(s) = \frac{K}{(1+sT_1)(1+sT_2)} \]  

(Type 0 system)

Solution: Given that:

\[ G(s)H(s) = \frac{K}{(1+sT_1)(1+sT_2)} \]

Put \( s = j\omega \)

\[ G(j\omega)H(j\omega) = \frac{K}{(1+j\omega T_1)(1+j\omega T_2)} \]

\[ |G(j\omega)H(j\omega)| = \frac{K}{\sqrt{1+\omega^2 T_1^2} \sqrt{1+\omega^2 T_2^2}} \]

\[ G(j\omega)H(j\omega) = -\tan^{-1} \omega T_1 - \tan^{-1} \omega T_2 \]

\[ \lim_{\omega \to 0} G(j\omega)H(j\omega) = K \]

\[ \lim_{\omega \to \infty} G(j\omega)H(j\omega) = 0 \]

\[ \lim_{\omega \to \infty} |G(j\omega)H(j\omega)| = 0 \]

\[ \lim_{\omega \to \infty} \angle G(j\omega)H(j\omega) = -180^\circ \]

---

Fig. 5.41.

The plot of \( G(j\omega)H(j\omega) \) is shown in Fig 5.41. The infinite semicircular arc of the Nyquist contour maps into origin. As the point \((-1 + jo)\) is not encircled by the plot

\[ N = 0 \quad P = 0 \quad Z = 0 \]

Hence, the system is stable.

Example 5.43. Using Nyquist criterion, determine the stability of the feedback system which has the following open loop transfer function.

\[ G(s)H(s) = \frac{K}{s^2(1+sT)} \]  

(Type 2' system)

Solution: Given that:

\[ G(s)H(s) = \frac{K}{s^2(1+sT)} \]

Put \( s = j\omega \)

\[ G(j\omega)H(j\omega) = \frac{K}{(j\omega)^2(1+j\omega T)} \]

Rationalizing the equation 5.39 and separating the real and imaginary part

\[ G(j\omega)H(j\omega) = \frac{K}{\omega^2(1+\omega T^2)} + j \frac{K}{\omega(1+\omega T^2)} \]

The Nyquist diagram is shown in the Fig (5.42). Because of the double pole at \( s = 0 \), a small semicircular detour at the origin should be made.
The point \((-1 + j\omega)\) is encircled twice. Hence \(N = 2\)
\[ P = 0 \]
\[ Z = 2 \]
Hence, the system is unstable.

Example 5.44. Use Nyquist criterion, determine whether the closed loop system having the following open loop transfer function is stable or not.

\[ G(s) H(s) = \frac{1}{s(1 + 2s)(1 + s)} \]

**Solution**: Given that

\[ G(s) H(s) = \frac{1}{s(1 + 2s)(1 + s)} \]

Put \(s = j\omega\)

\[ G(j\omega) H(j\omega) = \frac{1}{j\omega(1 + j2\omega)(1 + j\omega)} \]

Rationalizing the equation 5.41 and separate the real and imaginary part.

\[ G(j\omega) H(j\omega) = \frac{-3}{(1 + 4\omega^2)(1 + \omega^2)} - \frac{1 - 2\omega^2}{j\omega(1 + 4\omega^2)(1 + \omega^2)} \]

Equate the imaginary part to zero, we get the point of intersection on real axis

\[ 1 - 2\omega^2 = 0 \]
\[ \omega = 0.707 \]

\[ |G(j\omega) H(j\omega)|_{\omega = 0.707} = 0.66 \]

The Nyquist plot is shown in fig. 5.43.

Since, \(OA = 0.66\), the point \((-1 + j\omega)\) is not encircled

\[ N = 0 \]
\[ P = 0 \]
\[ Z = 0 \]

Fig. 5.43

Hence, the system is stable because no closed loop poles lie in the right half of s-plane.

Example 5.45. The open loop transfer function of a unity feedback system is given by

\[ G(s) = \frac{K}{s(1+sT_1)(1+sT_2)} \]

Derive an expression for gain \(K\) in terms of \(T_1, T_2\), and specified gain margin.

**Solution**: Given that

\[ G(s) = \frac{K}{s(1+sT_1)(1+sT_2)} \]

Put \(s = j\omega\)

\[ G(j\omega) = \frac{K}{j\omega(1+j\omega T_1)(1+j\omega T_2)} \]

Rationalizing the eqn 5.43 and separate the real and imaginary parts

\[ G(j\omega) H(j\omega) = \frac{-K(T_1 + T_2)}{(1 + \omega^2 T_1^2)(1 + \omega^2 T_2^2)} - \frac{1}{j\omega(1 + \omega^2 T_1^2)(1 + \omega^2 T_2^2)} \]

Equating the imaginary part to zero of eqn 5.44

\[ K(1 - \omega^2 T_1 T_2) = 0 \]

\[ \omega = \frac{1}{\sqrt{T_1 T_2}} \]

\[ G(j\omega) H(j\omega) = \frac{\frac{1}{\sqrt{T_1 T_2}}}{\frac{1}{T_1 T_2}} = \frac{KT_1 T_2}{T_1 + T_2} \]

\[ \text{Gain margin} \quad Gm = \frac{1}{a} \]

Here,

\[ a = \frac{KT_1 T_2}{T_1 + T_2} \]

\[ Gm = \frac{T_1 + T_2}{K(T_1 T_2)} \]

\[ K = \frac{1}{Gm} \left[ \frac{1}{T_1} + \frac{1}{T_2} \right] \]
This is the required expression. The polar plot is shown in the fig. 5.44.

\[ |G(j\omega)H(j\omega)| = \frac{1}{\omega \sqrt{1 + \omega^2 T_1^2 \sqrt{1 + \omega^2 T_2^2}}} \]

\[ \frac{G(j\omega)H(j\omega)}{\omega} = -90^\circ - \tan^{-1} \frac{\omega T_1}{1 - \omega^2 T_2} \]

\[ \lim_{\omega \to 0} |G(j\omega)H(j\omega)| = \infty \]

\[ \lim_{\omega \to \infty} \frac{G(j\omega)H(j\omega)}{\omega} = -90^\circ \]

\[ \lim_{\omega \to \infty} |G(j\omega)H(j\omega)| = 0 \]

\[ \lim_{\omega \to -\infty} G(j\omega)H(j\omega) = -270^\circ \]

Example 5.46. Consider a feedback system having the characteristic equation

\[ 1 + \frac{K}{(s+1)(s+1.5)(s+2)} = 0 \]

It is desired that all the roots of the characteristic equation have real parts less than -1. Use the Nyquist stability criterion to find the largest value of K, satisfying this condition.

Solution: Given that

\[ G(s)H(s) = \frac{K}{(s+1)(s+1.5)(s+2)} \]

Put \( s = -j\omega \)

\[ G(-j\omega)H(-j\omega) = \frac{K}{j\omega(0.5+j\omega)(1+j\omega)} \]

Rationalizing the equation 5.46 and separate the real and imaginary parts.

\[ G(-j\omega)H(-j\omega) = \frac{-1.5K}{(0.25\omega^2+i\omega^2)(1+\omega^2)} - j \frac{K(0.5-i\omega^2)}{\omega(0.25+i\omega^2)(1+i\omega^2)} \]

Equate the imaginary term to zero

\[ 0.5 - \omega^2 = 0 \]

\[ \omega = 0.707 \text{ rad/sec.} \]

\[ |G(-j\omega)H(-j\omega)|_{\omega=0.707} = 1.33K \]

As per condition given

\[ 1.33K < 1 \text{ or, } K < 0.7518 \]

\[ \text{Required largest value of } K \text{ is } 0.7518 \text{ Ans.} \]

Example 5.47. Use Nyquist criterion, determine whether the closed loop system having the following open loop transfer function is stable or not. If not how many closed loop poles lie in the right half s-plane?

\[ G(s)H(s) = \frac{1+4s}{s^2(1+s)(1+2s)} \]

(R.M.C. University Faisalabad 2002, 137)
2. Mapping of infinite semicircle
\[
\lim_{s \to \infty} \frac{1 + \text{Re}^s}{(1 + \text{Re}^s)(1 + 2 \text{Re}^s)} = 0 \Rightarrow -270^\circ
\]

3. Mapping of \( jo \)-axis from \( -\infty \) to \( 0 \), which is the mirror image of \( jo \)-axis from \( 0 \) to \( +\infty \).

4. Mapping small semicircle at origin
\[
\lim_{s \to -1} \frac{1 + \text{Re}^s}{(1 + \text{Re}^s)(1 + 2 \text{Re}^s)} = \omega (180^\circ \rightarrow 0^\circ \rightarrow -180^\circ)
\]

The complete Nyquist plot is shown in the figure 5.45.

Since, \( OA = -10 \cdot 64 \), hence \((1 + j0)\) encircles twice in clockwise direction.
\[
\begin{align*}
N & = 2 \\
P & = 0 \\
Z & = 2 \\
J & = 2
\end{align*}
\]

Hence, two roots of the characteristic equation lies in right half of \( s \)-plane. The closed loop system is unstable.

**Example 5.48.** A unity feedback system has open loop transfer function.

\[ G(s) = \frac{1}{s(1 + 2s)(1 + s)} \]

Sketch Nyquist plot for the system and therefore obtain the gain margin and the phase margin.

Solution: Given that \[ G(s) = \frac{1}{s(1 + 2s)(1 + s)} \]

Put \( s = jo \)

\[ G(jo) = \frac{1}{jo(1 + j2\omega)(1 + jo)} \]

Rationalizing the equation (5.47) and separate the real and imaginary parts.

\[ G(jo) = \frac{-3\omega^2}{\omega^2(1 + 4\omega^2)(1 + \omega^2)} - j \frac{\omega(1 - 2\omega)}{\omega^2(1 + 4\omega^2)(1 + \omega^2)} \]

Equate the imaginary part to zero
\[ 1 - 2\omega^2 = 0 \]
\[ \omega = \frac{1}{\sqrt{2}} = 0.707 \text{ rad/sec.} \]

\[ |G(jo)|_{\omega=0.707} = 0.667 \]

This is the point of intersection on real axis.

From eqn. 5.47
\[
|G(jo)| = \frac{1}{\omega^2 \sqrt{1 + 4\omega^2} \sqrt{1 + \omega^2}}
\]

\[ \text{Gain margin} = 20 \log_{10} \frac{1}{0.667} = 3.51 \]

For phase margin draw the unit circle which intersect the plot at A. Join the point 0 to A and measure the angle.

\[ \text{P.M} = 12 \]

**Example 5.49.** Draw the Nyquist plot for unity feedback control system, with open loop transfer function.

\[ G(s) = \frac{K(1-s)}{s+1} \]

Using Nyquist stability criterion, determine the stability of the closed loop system.

Solution: Given that

\[ G(s) = \frac{K(1-s)}{s+1} \]
Put $s = j\omega$

\[
G(j\omega) = \frac{K(1 - j\omega)}{(1 + j\omega)} |G(j\omega)| = K \frac{\sqrt{1 + \omega^2}}{\sqrt{1 + \omega^2}}
\]

\[
\frac{G(j\omega)}{\omega} = 0^o, \quad |G(j\omega)| = K
\]

\[
\frac{G(j\omega)}{\omega} = -90^o, \quad |G(j\omega)| = K
\]

\[
\frac{G(j\omega)}{\omega} = -180^o, \quad |G(j\omega)| = K
\]

The Nyquist plot is shown in fig. 5.47.

If $K > 1$ : No open loop pole on right half of s-plane $P = 0$.

Number of encirclement of point $(-1 + j0)$ $N = 1$

- $N = Z - P$
- $1 = Z - 0$
- $Z = 1$

Hence, the system is unstable.

If $K < 1$ :

- $P = 0$
- $N = 0$
- $Z = 0$

Hence, the system is stable.

Example 5.50. Using Nyquist criterion investigate the stability of a closed loop control system whose open loop transfer function is given below

\[
G(s)H(s) = \frac{K}{s(1 + sT_1)(1 + sT_2)}
\]

Solution : Given that

\[
G(s)H(s) = \frac{K}{s(1 + sT_1)(1 + sT_2)}
\]

Put $s = j\omega$

\[
G(j\omega)H(j\omega) = \frac{K}{j\omega(1 + j\omega T_1)(1 + j\omega T_2)}
\]

Rationalizing the equation 5.48 and separate the real and imaginary part

\[
G(j\omega)H(j\omega) = \frac{K(T_1 + T_2)}{(1 + \omega^2 T_1^2)(1 + \omega^2 T_2^2)} - j \frac{K(1 - \omega^2 T_1 T_2)}{(1 + \omega^2 T_1^2)(1 + \omega^2 T_2^2)}
\]

The value of $\omega^2$ at the point of intersection of plot with real axis can be obtained by equating the imaginary part to zero.

\[
\frac{K(1 - \omega^2 T_1 T_2)}{\omega(1 + \omega^2 T_1^2)(1 + \omega^2 T_2^2)} = 0
\]

\[
\omega = \sqrt{\frac{T_1 T_2}{T_1 + T_2}}
\]

\[
\omega = \frac{1}{\sqrt{T_1 T_2}}
\]

For stable system

\[
K < \frac{T_1 + T_2}{T_1 T_2}
\]

\[
K < \frac{T_1 + T_2}{T_1 T_2}
\]

(a) When the gain $K$ is less than $\frac{T_1 + T_2}{T_1 T_2}$, the point $(-1 + j0)$ is not encircled by $G(j\omega)H(j\omega)$ plot and the system is stable.

(b) When $K = \frac{T_1 + T_2}{T_1 T_2}$, the plot passes through $(-1 + j0)$ which indicates the system has roots on $j\omega$-axis. Hence, the system is marginally stable.

(c) When $K > \frac{T_1 + T_2}{T_1 T_2}$, the plot encircles the point $(-1 + j0)$. Hence the system is unstable.

The plot is shown in fig. 5.48.

Example 5.51. Consider the transfer function

\[
G(s)H(s) = \frac{60}{(s + 1)(s + 2)(s + 5)}
\]

Using Nyquist stability criterion determine whether the closed loop system is stable or not.

Solution : Given that

\[
G(s)H(s) = \frac{60}{(s + 1)(s + 2)(s + 5)}
\]

Put $s = j\omega$

\[
G(j\omega)H(j\omega) = \frac{60}{(1 + j\omega)(2 + j\omega)(5 + j\omega)}
\]

Rationalizing the equation 5.50 and separate the real and imaginary parts.

\[
G(j\omega)H(j\omega) = \frac{60(10 - 8\omega^2)}{(1 + \omega^2)(4 + \omega^2)(25 + \omega^2)} - j \frac{60(17\omega - \omega^3)}{(1 + \omega^2)(4 + \omega^2)(25 + \omega^2)}
\]

Equate the real part to zero.

\[
\frac{60(10 - 8\omega^2)}{(1 + \omega^2)(4 + \omega^2)(25 + \omega^2)} = 0
\]

\[
\omega = 1.25 \text{ rad/sec}
\]
Determine the value of $K$ so that the system may be stable with
a. gain margin equal to 6 dB
b. $P.M.$ equal to $45^\circ$

Solution:

- $G.M. = 20\log_{10}\left(\frac{1}{a}\right)$

Where '$a$' is the point of intersection of Nyquist plot on negative real axis.

- $a = 0.501$

Since,

$$G(s) = \frac{K}{s(s + 2)(s + 10)}$$

Put $s = ja$

$$K = \frac{1}{j\omega(2 + j\omega)(10 + j\omega)}$$  \hspace{1cm} \text{(5.51)}

Rationalizing the equation 5.51 and separate the real and imaginary parts.

$$G(j\omega) = \frac{-12K\omega^2}{\omega^2(4 + \omega^2)(100 + \omega^2)} - j\frac{K(20\omega - \omega^3)}{\omega^2(4 + \omega^2)}$$ \hspace{1cm} \text{(5.52)}

To get the point of intersection on real axis, equate the imaginary part to zero.

$$\omega = 1.25$$

$$\omega = 4.47 \text{ rad/sec.}$$

$$\left|G(j\omega)\right|_{\omega = 4.47} = -0.0041 \text{ K}^*$$

Put $\omega = 4.47 \text{ rad/sec.}$ in eqn no. 5.52.

$$0.0041K = 0.501$$

$$K = 122$$

Example 5.52. The open loop transfer function of a control system is given by

$$G(s) = \frac{K}{s(s + 2)(s + 10)}$$

The mapping is as follows:

1. Part $OA$ in the $s$-plane is the $jo\omega$-axis from $\omega = 0$ to $\omega = \infty$ and maps into the polar plot $A'O$ in $GH$-plane.

2. The infinite semicircle $ABC$ maps into origin in $GH$-plane.

3. Part $CO$ which is the negative part of $jo\omega$ axis, maps into $OA'$ From Nyquist plot, the point $(-1 + j0)$ is not encircled by the plot

$$N = 0$$

No pole in the right half

$$P = 0$$

$$N = Z - P$$

$$Z = 0$$

Therefore, no zero in right half and hence the system is stable.
5.17. ANALYSIS OF STABILITY BY LYAPUNOV'S DIRECT METHOD

5.17.1 Introduction

This method is useful for determining the stability of the non-linear system. This method is also applicable for linear systems. In this method, the stability of the system can be determined without actually solving the differential equation, that’s why it is called direct method.

The basic concept of this is that the energy stored in a stable system cannot increase with time. If we obtain a function $V(x)$ which is suitable function of energy and examine it as a function of time, then for the stability of the function of this type approaches zero as the time approaches infinity.

5.17.2 The Concept of Definiteness

Let $V(x)$ be a real scalar function.

The scalar function $V(x)$ is said to be positive definite if the function $V(x)$ has always positive sign in the given region about the origin, except only at the origin where it is zero, or in other words, the scalar function $V(x)$ is positive definite in the given region if $V(x) > 0$ for all non-zero states $x$ in the region and $V(0) = 0$.

The scalar function $V(x)$ is said to be negative definite if the function $V(x)$ has always negative sign in the given region about the origin, except only at the origin where it is zero or a scalar function $V(x)$ is said to be negative definite if $-V(x)$ is positive definite.

For example:

$v(x) = x_1^2 + x_2^2 + \cdots + x_n^2$, is positive definite

$v(x) = -x_1^2 - (x_1 + x_2)^2$, is negative definite

$v(x) = -x_1^2 - (x_1 + 2x_2)^2$, is negative definite

5.17.3 Lyapunov Stability Theorem.

THEOREM 5.17.4. Suppose a system is described by

\[ \dot{x} = f(x) \]

If there exists a scalar function $V(x)$ which is real, continuous and has continuous first partial derivatives with

$V(x) > 0$ for $x \neq 0$

$V(0) = 0$

then the system is asymptotically stable.

$V(x)$ is the Lyapunov function.

Example 5.33. The system is given by

\[ \dot{x}_1 = x_1 \]
\[ \dot{x}_2 = -x_1 - x_2^2 \]

Investigate the system by Lyapunov’s method using $V = x_1^2 + x_2^3$.

5.17.4 The Direct Method of Lyapunov and Linear System

Let the system is given by

\[ \dot{x} = Ax \]

Select the Lyapunov function as

$V(x) = x^TPx$ \hspace{1cm} (5.54)

$\dot{V}(x) = \dot{x}^TPx + x^TP\dot{x}$ \hspace{1cm} (5.55)
Substitute the value of $x$ from 5.53

\[
\dot{V}(x) = (Ax)^T P x + x^T P A x = x^T A^T P x + x^T P A x
\]

\[
= x^T [A^T P + PA] x
\]

\[
\dot{V}(x) = x^T Q x
\]

where,

\[
Q = [A^T P + PA]
\]

where $Q$ is a positive definite matrix. Select a positive definite $Q$ and find the $P$ from equation (5.57) if $P$ is a positive definite then the system will be stable.

**PROCEDURE. Step 1:** Select $Q$ as a positive definite

**Step 2:** Obtain $P$ from the equation 5.57 for this we will have to solve \( \frac{n(n+1)}{2} \) number of equations, where 'n' is the order of matrix $A$.

**Step 3:** Using the Sylvester theorem (Sylvester theorem is given at the end of this chapter) determine the definiteness of $P$. If $P$ is positive definite the system is stable otherwise unstable. Generally, $Q$ can be taken as identity matrix.

**Example 5.55. Determine the stability of the system**

\[
\dot{x} = Ax
\]

\[
A = \begin{bmatrix}
1 & -2 \\
1 & -4
\end{bmatrix}
\]

**Solution:** Given that

\[
Q = \begin{bmatrix}
1 & 0 \\
0 & -1
\end{bmatrix}
\]

Select

\[
P = \begin{bmatrix}
11 & 6 \\
6 & 11
\end{bmatrix}
\]

we know that

\[
Q = [A^T P + PA]
\]

\[
\begin{bmatrix}
-1 & 0 \\
0 & -1
\end{bmatrix} = \begin{bmatrix}
1 & 1 \\
-2 & 4
\end{bmatrix} \begin{bmatrix}
P_{11} & P_{12} \\
P_{21} & P_{22}
\end{bmatrix} + \begin{bmatrix}
P_{11} & P_{12} \\
P_{21} & P_{22}
\end{bmatrix} \begin{bmatrix}
1 & -2 \\
-1 & 1
\end{bmatrix}
\]

\[
\begin{bmatrix}
-1 & 0 \\
0 & -1
\end{bmatrix} = \begin{bmatrix}
2P_{11} + 2P_{12} - 2P_{11} + 5P_{12} + 2P_{22} \\
-2P_{11} + 5P_{12} + 2P_{22} - 4P_{12} - 8P_{22}
\end{bmatrix}
\]

Take $p_{11} = p_{22}$, this is because solution matrix $P$ is known to be a positive definite real symmetric matrix for a stable system.

\[
-2p_{11} + 2p_{12} = -1
\]

\[
-2p_{11} - 5p_{12} + p_{22} = 0
\]

\[
-4p_{12} - 8p_{22} = -1
\]

Solving the equation 5.60 and 5.61 we get

\[
7p_{12} - p_{22} = -1
\]

Now solving (5.62) and (5.63) we get

\[
p_{12} = \frac{1}{7} \quad p_{22} = \frac{-1}{7}
\]

From (5.60) we get

\[
p_{11} = \frac{23}{60}
\]

From (5.62)

\[
p_{11} = 1
\]

\[
P = \begin{bmatrix}
\frac{23}{60} & \frac{-7}{60} \\
\frac{-7}{60} & \frac{11}{60}
\end{bmatrix}
\]

Hence, the origin of the system under consideration is asymptotically stable in-the-large.

**Example 5.56. Consider a non-linear system described by the equations**

\[
x_1 = -3x_1 + x_2
\]

\[
x_2 = -x_1 - x_2 - x_3
\]

**Investigate the stability of the equilibrium state.**

**Solution:** Let

\[
V(x) = x_1^2 + x_2^2
\]

\[
\dot{V}(x) = x_1 \frac{\partial V}{\partial x_1} + x_2 \frac{\partial V}{\partial x_2}
\]

\[
\dot{V}(x) = 2x_1 \dot{x}_1 + 2x_2 \dot{x}_2
\]

Substitute the values of $\dot{x}_1$ and $\dot{x}_2$

\[
\dot{V}(x) = 2x_1 (x_1 - 2x_2) + 2x_2 (x_1 - x_2 - x_3)
\]

\[
\dot{V}(x) = -6x_1^2 - 2x_2^2 - 2x_3^2 = \text{negative definite}
\]

Hence, the system is asymptotically stable in-the-large.

**Example 5.57. Check the stability of the system described by**

\[
x_1 = x_2
\]

\[
x_2 = -x_1 - x_2^2 - x_3
\]

**Solution:** Let

\[
V(x) = x_1^2 + x_2^2
\]

\[
\dot{V}(x) = x_1 \frac{\partial V}{\partial x_1} + x_2 \frac{\partial V}{\partial x_2}
\]

\[
\dot{V}(x) = 2x_1 \dot{x}_1 + 2x_2 \dot{x}_2
\]

Substitute the values of $x_1$ and $x_2$

\[
\dot{V}(x) = 2x_1 x_2 + 2x_2 (x_1 - x_2 - x_3)
\]

\[
\dot{V}(x) = -2x_1 x_2^2 = \text{negative definite}
\]

Hence, the system is asymptotically stable.

**Example 5.58. Examine the stability of the system described by the equation**

\[
x_1 = x_2
\]

\[
x_2 = -6x_1 - 5x_2
\]
Solution:

\[ V(x) = x_1^2 + x_2^2 \]

\[ \dot{V}(x) = \frac{\partial V}{\partial x_1} \dot{x}_1 + \frac{\partial V}{\partial x_2} \dot{x}_2 \]

\[ \dot{V}(x) = 2x_1 \dot{x}_1 + 2x_2 \dot{x}_2 \]

Put the values of \( \dot{x}_1 \) and \( \dot{x}_2 \):

\[ \dot{V}(x) = 2x_1(x_1) + 2x_2(-6x_1 - 5x_2) \]

\[ = -10x_1x_2 - 10x_2^2 = \text{negative definite} \]

Hence, the system is asymptotically stable.

**THEOREM 5.17.** If theorem 5.16.4 is satisfied and in addition

\[ V(x) \to \infty \]

as \( ||x|| \to \infty \),

then the system is asymptotically stable-in-the large at origin.

**5.18. SYLVESTER'S THEOREM**

This theorem states that the necessary and sufficient conditions that the quadratic form \( V(x) \) is positive definite are that all the successive principal minors of \( P \) be positive i.e.

\[ p_{11} > 0 \]
\[ p_{11} p_{22} - p_{12}^2 > 0 \]
\[ p_{11} p_{22} p_{33} - p_{12} p_{23}^2 - p_{13} p_{22} > 0 \]

(a) \( V(x) \) is negative definite if \( -V(x) \) is positive definite.

(b) \( V(x) = x^T P x \) is positive semidefinite if \( P \) is singular and all the principal minors are non negative.

Example 5.59. Determine whether the following quadratic form is negative definite.

\[-Q = x_1^2 + 3x_2^2 + 11x_3^2 - 2x_1 x_2 + 4x_2 x_3 + 2x_1 x_3 \]

Solution:

\[ [x_1, x_2, x_3] \begin{bmatrix} 1 & -1 & 1 \\ -1 & 3 & 2 \\ 1 & 2 & 11 \end{bmatrix} \]

Note: The middle matrix is found as the coeff of

\[ \begin{bmatrix} x_1^2 & x_1 x_2 & x_1 x_3 \\ x_1 x_2 & x_2^2 & x_2 x_3 \\ x_1 x_3 & x_2 x_3 & x_3^2 \end{bmatrix} \]

Now, \( p_{11} = 1 > 0 \)

\[ \begin{bmatrix} 1 & -1 \\ -1 & 3 \end{bmatrix} = 2 > 0, \quad \begin{bmatrix} 1 & -1 & 1 \\ -1 & 3 & 2 \\ 1 & 2 & 11 \end{bmatrix} = 11 > 0 \]

Hence \(-Q\) is positive definite therefore quadratic form is negative definite.
4. By this method we can determine the point of intersection of root locus with negative axis.
5. For unstable system it gives the number of roots of characteristic equation having negative real part.

Limitations of Routh-Harwitz criterion:
1. This is applicable only for linear systems.
2. It does not provide the exact location of closed loop poles in left or right half of s-plane.
3. It is valid only for real coefficients of the characteristic equation.
4. The location of the roots of the characteristic equation when gain is varied from zero to infinity is called root locus.
5. The root loci start from open loop poles and terminate on open loop zeros or on infinity.
6. The asymptotes are the guide for the branches approaching to infinity.
7. The point of intersection of asymptotes with real axis is called centroid.
8. Angle conditions used to test whether a point in s-plane is on the root locus or not.
9. Magnitude conditions give the value of K corresponding to any point which is on the root locus.

Nyquist stability criterion states that a closed loop system is stable if the contour GH of the open loop transfer function $G(s)H(s)$ corresponding to the Nyquist contour in $s$-plane encircles the point $(-1+0j)$ in counterclockwise direction and the number of counterclockwise encirclement about $(-1+0j)$ equals the number of poles of $G(s)H(s)$ in the right half of s-plane.

The closed loop system is stable if the contour GH of $G(s)H(s)$ does not pass through or does not encircle $(-1+0j)$ point i.e., net encirclement is zero.

Lyapunov stability Theorem
If a system is described by

$$\dot{x} = f(x)$$

If there exists a scalar function $V(x)$ which is real, continuous and has continuous first partial derivatives with $V(x) > 0$ for $x \neq 0$ $V(0) = 0$ $V(x) < 0$ for all $x \neq 0$ then the system is asymptotically stable.

**EXERCISE**

5.1. Sketch the root locus for open loop transfer function $\frac{K(s+1)}{s(s^2+2s+2)}$.

5.2. Consider the closed loop feedback control system shown in Fig. 5.50. Using Routh-Harwitz criterion determine the range of K for which the system is stable. Find also the number of roots of the characteristic equation that are in the right half of s-plane for $K = 0.5$.

5.3. Sketch the root locus for the given system (Fig. 5.51). Determine the value of K at which the system becomes unstable.

**ANSWERS**

5.2. Stable for $0 < K < 4.67$.

5.3. For $K = 0.5$, two roots in right half.

5.4. $\tau < 1.49$, $\omega = 0.316$ rad/sec.

5.5. Phase cross over frequency = 4.472 rad/sec.

5.6. Stable if $K > 2$.

5.7. $K > 0$ and $T > K/B$.

For second part $200 + 10T - K > 0$ $9 + T + K > 0$. 

5.8. Sketch the root locus plot for the characteristic equation $\frac{s^2 + 30s + 200}{s(s+2)}$.

5.9. The characteristic equation of a system is given by $s^4 + 20s^3 + 15s^2 + 2s + K = 0$.

5.10. Determine range of K for system to stable.

5.11. Can system be marginally stable? If so, find the required value of K and the frequency of sustained oscillation.

5.12. Apply Routh's criterion, determine the number of roots that lie in the right half-plane of the following characteristic equation $s^4 + 6s^3 + 12s^2 + 13s - 16s - 24 = 0$ $s^4 + 2s^2 + 4s + 8 = 0$.

5.13. The characteristic equation for a control system is $s(s^2 + 5s + 20) + K(s + 2) = 0$.

5.14. Sketch the Nyquist plot for $0 < \omega < \infty$ and find the phase crossover frequency for $G(s)H(s) = \frac{200K}{s+2}(s+10)$.

5.15. For open transfer function $G(s)H(s) = \frac{K(s+2)}{(s+1)(s-3)}$, sketch the Nyquist plot for determine whether the system is stable or not.

5.16. Point out the limitations of Routh Hurwitz stability criterion. The open loop transfer function of a unity feedback control system is given by $G(s) = \frac{K}{s(s^2 + 8s + 7)}$. Using Routh-Harwitz criterion determine the values of K & T which corresponds to a stable system.

5.17. Also determine the new values of K & T if it is required that all the roots of the characteristic equation lie in the region to the left of the line s = -1.

5.18. (LPSC 1974) For second part modified the characteristic equation by putting $s = z + 1$ in the characteristic equation & then apply Routh Hurwitz criterion.
SEMIOBJECTIVE TYPE QUESTIONS

(i) Define stability.
(ii) State Routh Hurwitz criterion.
(iii) State necessary but not sufficient conditions for stability.
(iv) Define marginally stable system.
(v) What are the two special cases which occur in Routh's table? How are those cases handled?

(vi) Define relative stability.
(vii) What is root locus?
(viii) What is breakaway point?
(ix) Write short note on Nyquist criterion.
(x) Write short note on mapping.
(xi) State Sylvester's Theorem.
(xii) State advantages of Routh Hurwitz criterion.
(xiii) State limitations of Routh Hurwitz criterion.
(xiv) State Lyapunov stability theorem.
(xv) Write a note on angle and magnitude condition of root locus.
(xvi) Explain the term (a) asymptotes (b) centroid.
(xvii) Write a short note on effects of adding pole and zero on the root locus.
(xviii) State the rule for obtaining the breakaway point in the root loci.
(xix) How to obtain the open gain at any point on the root loci?
(xx) How location of roots of the characteristic equation are related to stability?
(xxi) How will you obtain the angle of departure and angle of arrival of root loci?
(xxii) How will you determine the absolute value of $K$ at any point $S_i$ on the root loci?

Chapter 6

Compensation Techniques

6.1. INTRODUCTION

Sometimes it is necessary to compensate an unstable system to make it stable or it may be necessary to improve the existing system to satisfy or to meet the required specifications. The system performance is given by time response and frequency response.

For the design of systems in s-domain, introduction of poles and zeros at suitable places will give satisfactory performance. Main requirement of the control system is accuracy and stability. For greater accuracy of a system steady state error should be small, but to reduce the steady state error the gain of the amplifier must be increased. However, the gain of the amplifier must be increased, the steady state error will also increase and stability will decrease. But we are interested in both accuracy and stability. This can be done by connecting a circuit between error detector and plant known as compensation.

Now consider a transfer function

$$G(s) = \frac{K}{s(s+1)(s+2)}$$

It is required that velocity constant $K_v$ corresponding to steady state error should be 10 sec$^{-1}$.

We know that

$$K_v = \lim_{s \to 0} sG(s)$$

$$K_v = \lim_{s \to 0} s \frac{K}{s(s+1)(s+2)} = K/2$$

$$10 = K/2$$

$$K = 20$$

Therefore for accuracy $K$ should be equal to 20.

Now equation 6.1 becomes

$$G(s) = \frac{20}{s(s+1)(s+2)}$$

The characteristic equation of 6.1 will be

$$1 + G(s) H(s) = 0$$

$$1 + \frac{K}{s(s+1)(s+2)} = 0$$

$$s^3 + 3s^2 + 2s + K = 0$$
6.2. TYPES OF COMPENSATION

Depending upon the location of compensating network, following are the types of compensation:

1. **Series compensation**: When a compensating network is inserted in the forward path, this is called series or cascade compensation, as shown in fig. 6.2(a).

2. **Feedback compensation**: When a compensator is inserted in feedback path, this is called feedback compensation, as shown in fig. 6.2(b).

3. **Load compensation**: A combination of series and feedback compensation is called load compensation or combined cascade and feedback compensation as shown in fig. 6.2(c).

The compensators may be electrical, mechanical, pneumatic or hydraulic type. Mostly electrical networks are used. It is easy to design R.C. filters. For the design of compensation network mainly transfer function approach is used. The simplest networks are lead, lag and lag-lead networks.

6.3. DESIGN OF COMPENSATION USING BODE'S PLOT

Bode diagram approach is commonly used because of its simplicity and performance specifications are given in frequency domain.

6.3.1. Phase Lead Network

From circuit diag. 

\[
\begin{align*}
\frac{E_s(s)}{E_i(s)} &= \frac{R_2}{R_1 + R_2} \left[ 1 + \frac{R_1 R_2 C}{R_1 + R_2} \right] \\
\frac{E_s(s)}{E_i(s)} &= \frac{1}{\alpha \left[ 1 + \frac{1}{1 + a T} \right]} \\
\end{align*}
\]
6.3.1.2. Bode Plot for Phase Lead Network

From equation 6.8:
\[
\frac{aE_2(s)}{E_1(s)} = \frac{1 + aTs}{1 + Ts}
\]
\[
\phi = \text{Arg} \left[ \frac{aE_2(s)}{E_1(s)} \right] = \tan^{-1} aT \omega - \tan^{-1} \omega T
\]
or,
\[
\tan \phi = \frac{aT \omega - T \omega}{1 + (aT \omega) (T \omega)}
\]
\[
\tan \phi = \frac{T \omega (a-1)}{1 + aT \omega}
\]

For maxm. Value of \( \phi \),
\[
\frac{d\phi}{d\omega} = 0
\]

We get
\[
\omega_m = \frac{1}{T \sqrt{a}}
\]

6.3.1.3. Calculation of \( \phi_m \) and \( \omega_m \)

From equation 6.8:

\[
\tan \phi_m = \frac{a - 1}{2 \sqrt{a}}
\]
\[
\tan \phi_m = \frac{a - 1}{a + 1}
\]
\[
\sin \phi_m = \frac{a - 1}{1 - \sin \phi_m}
\]

From eqn. 6.12 we can calculate the value of 'a':
\[
a = \frac{1 + \sin \phi_m}{1 - \sin \phi_m}
\]

**DESIGN PROCEDURE FOR PHASE LEAD COMPENSATION**

Step 1: The magnitude and phase Vs frequency curves are plotted for \( G(s) \) of the uncompensated system, with gain constant \( K \) set according to steady state error requirement.

Step 2: From the Bode's plot determine the phase margin and gain margin.

Let
\[
\phi = \text{phase margin of uncompensated system}
\]
\[
\phi_m = \text{specified phase margin}
\]
\[
\epsilon = \text{margin of safety}
\]

then, \( \phi_m = \phi - \phi + \epsilon \)

If \( \phi_m > 60^\circ \), two identical network each contributing maximum lead of \( \phi_m / 2 \) are used.

Step 3: Use equation 6.12 calculate the value of 'a'.

Step 4: Calculate \( \omega_m \) at which uncompensated system will have a gain equal to
\[
-10 \log_a \text{ (the phase lead network causes an attenuation of } \frac{1}{2} \text{ (20log}_a \text{ a) = })
\]
\[
-10 \log_a \text{ db at frequency } \omega_m
\]

Step 5: Once 'a' is determined, calculate the value of 'T' from \( \omega_m = \frac{1}{\sqrt{a} T} \)

Step 6: Transfer function of phase lead network is determined from the values of 'a' & 'T'.

Step 7: Draw the Bode's plot of the compensated system and check that all performance specifications are met or not. If not, a new value of \( \phi_m \) must be estimated.

6.3.2. Effect of Phase Lead Network

1. The velocity constant is usually increased.
2. The slope of the magnitude curve is reduced at the gain cross over frequency, with the result relative stability improve.
3. Phase margin increased.
4. The bandwidth increased.
5. The response is faster.

Example 6.1. Design a cascade compensation for a system whose transfer function is

\[ G(s) = \frac{K}{s(1 + 0.1s)(1 + 0.001s)} \]

It will fulfill the following specifications:
Phase margin \( \geq 45^\circ \)
Velocity constant \( K_p = 1000 \text{ sec}^{-1} \)

Solution: Step 1:
\[ K_p = \lim_{s \to 0} G(s) = \lim_{s \to 0} s \cdot \frac{K}{s(1 + 0.1s)(1 + 0.001s)} \]
\[ K_p = K = 1000 \]
\[ G(s) = \frac{1000}{s(1 + 0.1s)(1 + 0.001s)} \]

Step 2: Draw the Bode’s plot for the transfer function given by the eqn 6.14.
Two corner frequencies are \( 1/0.1 = 10 \text{ rad/sec} \) and \( 1/0.001 = 1000 \text{ rad/sec} \).

<table>
<thead>
<tr>
<th>( \omega )</th>
<th>( \text{Arg}(1000) )</th>
<th>( \text{Arg}(0 + j\omega) )</th>
<th>( \text{Arg}(1 + 0.1j\omega) )</th>
<th>( \text{Arg}(1 + j0.001\omega) )</th>
<th>Resultant ( \psi_1 + \psi_2 + \psi_3 + \psi_4 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0</td>
<td>-90°</td>
<td>-5.7°</td>
<td>-0.06</td>
<td>-95.7°</td>
</tr>
<tr>
<td>5</td>
<td>0</td>
<td>-90°</td>
<td>-26.5°</td>
<td>-0.28</td>
<td>-115.6°</td>
</tr>
<tr>
<td>10</td>
<td>0</td>
<td>-90°</td>
<td>-45°</td>
<td>-0.57</td>
<td>-135.3°</td>
</tr>
<tr>
<td>50</td>
<td>0</td>
<td>-90°</td>
<td>-76.6°</td>
<td>-2.86°</td>
<td>-171.46°</td>
</tr>
<tr>
<td>100</td>
<td>0</td>
<td>-90°</td>
<td>-84.2°</td>
<td>-5.71°</td>
<td>-179.9°</td>
</tr>
<tr>
<td>150</td>
<td>0</td>
<td>-90°</td>
<td>-86.2°</td>
<td>-8.5°</td>
<td>-184°</td>
</tr>
<tr>
<td>200</td>
<td>0</td>
<td>-90°</td>
<td>-87.13°</td>
<td>-11.3°</td>
<td>-188.43°</td>
</tr>
<tr>
<td>500</td>
<td>0</td>
<td>-90°</td>
<td>-88.85°</td>
<td>-26.56°</td>
<td>-205.41°</td>
</tr>
</tbody>
</table>

Step 3: From Bode plot:
Phase margin available \( \phi = 0° \)
Specified phase margin \( \phi_s = 45° \)
Margin of safety \( \varepsilon = 5° \)
\[ \phi_m = 45° - 0° + 5° = 50° \]

Step 4: Calculation of \( a' \)
\[ \sin \phi_m = \frac{a - 1}{a + 1} \]
\[ \sin 50° = \frac{a - 1}{a + 1} \]
\[ a = 7.51 \]

Step 5: Calculation of \( \omega_m \)
Zero frequency attenuation = \( -10 \text{log}_a \)
\[ a = 10 \log 7.51 = 8.75 \text{ db} \]
At the gain of \( -8.75 \text{ db} \) draw a line on magnitude curve, this will give \( \omega_m \) (new gain cross over frequency).
\[ \omega_m = 170 \text{ rad/sec} \] (from Bode plot)

6.3.3. Limitations of Phase Lead Networks
For getting large phase margin we required large value of \( a' \) which increases the bandwidth, this may increase the transmission of noise. In practice the value of \( a' \) should not be greater than 15. If the large phase lead is required two or more cascade compensation should be used.

If the phase lead is required at a new gain crossover frequency, phase lead compensation becomes ineffective because the additional phase lead at new gain crossover frequency is added to the a much smaller phase angle than that at the old gain cross over frequency. The desired phase margin can be achieved only with large value of \( a' \) which is not desirable.
6.1.4 Phase Lag Network

A simple and commonly used network is shown in Fig. 6.4.

Transfer Function

Apply KVL in both mesh

\[ e_1 = \frac{R_1 i + R_2 i + \frac{1}{c} \int i \, dt}{c} = (6.15) \]

\[ e_0 = \frac{R_2 i + \frac{1}{c} \int i \, dt}{C} = (6.16) \]

Take Laplace transform of eqn. 6.15 & 6.16

\[ E_1(s) = R_1 I(s) + R_2 I(s) + \frac{1}{C} I(s) = (6.17) \]

\[ E_0(s) = R_2 I(s) + \frac{1}{C} I(s) = (6.18) \]

From eqns. (6.17) & (6.18)

\[ \frac{E_0(s)}{E_1(s)} = \frac{1 + R_2 Cs}{(R_1 + R_2)Cs + 1} = (6.19) \]

Equation 6.19 is the transfer function of lag network

Put

\[ aT = R_2 C \]

\[ a = \frac{R_2}{R_1 + R_2} \]

\[ T = (R_1 + R_2)C \]

then

\[ \frac{E_0(s)}{E_1(s)} = \frac{1 + aTs}{1 + Ts} \quad a < 1 = (6.20) \]

Pole-Zero Plot

The lag network has a pole at \( s = -\frac{1}{T} \) and zero at \( s = -\frac{1}{aT} \). From pole-zero plot shown in Fig. 6.4a, the pole is nearer to the imaginary axis as compared to zero, hence the effect of pole is dominant, therefore introduction of phase lag network in forward path reduces the phase shift.

Bode Plot for Phase Lag Network

From Bode plot, the phase lag network provides an attenuation of \( 20\log_{10}a \) at high frequencies. The lag network allows to pass low frequencies and high frequencies are attenuated.
DESIGN PROCEDURE FOR PHASE LAG NETWORK

**Step 1:** The magnitude and phase V/s frequency curves (Bode plot using asymptotic approximation) are plotted for G(s) of the uncompensated system, with gain constant K set according to steady-state error requirement.

**Step 2:** From the Bode plot, determine the phase margin of the uncompensated system.

**Step 3:** If $\phi_e$ = specified phase margin, $\varepsilon$ = margin of safety, then $\phi = \phi_e + \varepsilon$.

**Step 4:** Determine the frequency corresponding to the required phase margin from the phase curve. This frequency is new gain crossover frequency ($\omega_m'$).

**Step 5:** The magnitude curve is brought down to 0 db at the new gain crossover frequency when phase margin is satisfied, the phase lag network must provide the amount of attenuation equal to the value of magnitude curve at $\omega_m'$.

$$|G(\omega_m')| = 20 \log a$$

or,

$$a = 10^{-\frac{\text{Arg}(G)}{20}}$$

**Example 6.3:** Design a suitable lag compensating network for

$$G(s) = \frac{K}{s(s+2)(s+20)}$$

to meet the following specifications:

- $K_p = 20 \text{ sec}^{-1}$
- P.M. $\geq 35^\circ$

**Solution:**

**Step 1:**

$$K_p = \lim_{s \to 0} sG(s) = \lim_{s \to 0} \frac{K}{s(s+2)(s+20)} = 20$$

$$K = 800$$

$$\therefore G(s) = \frac{800}{s(s+2)(s+20)} = \frac{20}{s(1+0.5s)(1+0.05s)} \quad (6.21)$$

**Step 2:** Draw the Bode plot for $\omega_m$.

There are two corner frequencies:

- $\omega_1 = 1/0.5 = 2 \text{ rad/sec}$.
- $\omega_2 = 1/0.05 = 20 \text{ rad/sec}$.

<table>
<thead>
<tr>
<th>$\omega$</th>
<th>$-\text{Arg}(G)$</th>
<th>$-\text{Arg}(1 + j0.5\omega)$</th>
<th>$-\text{Arg}(1 + j0.05\omega)$</th>
<th>Resultant</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.1</td>
<td>-90°</td>
<td>-2.86°</td>
<td>-0.286°</td>
<td>-93.14°</td>
</tr>
<tr>
<td>0.2</td>
<td>-90°</td>
<td>-5.71°</td>
<td>-0.571°</td>
<td>-96.28°</td>
</tr>
<tr>
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<td>-90°</td>
<td>14.04°</td>
<td>-1.43°</td>
<td>105.47°</td>
</tr>
<tr>
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<td>-2.86°</td>
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<td>140.71°</td>
</tr>
<tr>
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<td>-90°</td>
<td>68.2°</td>
<td>-14.04°</td>
<td>172.24°</td>
</tr>
<tr>
<td>6</td>
<td>-90°</td>
<td>71.56°</td>
<td>-16.7°</td>
<td>178.26°</td>
</tr>
<tr>
<td>7</td>
<td>-90°</td>
<td>74.05°</td>
<td>-19.29°</td>
<td>183.34°</td>
</tr>
<tr>
<td>8</td>
<td>-90°</td>
<td>78.96°</td>
<td>-21.8°</td>
<td>187.76°</td>
</tr>
<tr>
<td>10</td>
<td>-90°</td>
<td>78.69°</td>
<td>-26.57°</td>
<td>195.26°</td>
</tr>
</tbody>
</table>
Step 3: P.M available = 0°
   Specified phase margin = 35°
   Margin of safety ε = 5°
   ϕ = 35° + 5° = 40°
   For desired P.M. the gain crossover frequency must be shifted to a lower value and this is possible by using phase lag compensation.

Step 4: Required phase margin is available at ω = 2 rad/sec.
   This is new gain crossover frequency. Draw a line at -140° the point where it cuts the phase curve project it to the magnitude curve.
   From Bode plot gain available at ω_m = (2 rad/sec) = 20 db

Step 5:
   \[ |G(jω_m)| = -20 \log a \]
   20 = 20 loga
   a = 0.1

Step 6: Calculation of \( T \):
   \[ \frac{1}{ω_m} = \frac{2}{10} \]
   \[ T = 50 \]

Step 7: Calculation of \( G_C(s) \)
   \[ G_C(s) = \frac{1 + aTs}{1 + Ts} \]
   \[ G_C(s) = \frac{1 + 5s}{1 + 50s} \]

Step 8: Overall transfer function after compensation
   \[ G(s) = \frac{20(1 + 5s)}{s(1 + 0.5s)(1 + 0.05s)(1 + 50s)} \]

Step 9: Draw the Bode plot for \( ω_m^2 = 6.22 \).
   Corner frequencies are
   \( ω_1 = 0.02 \text{ rad/sec} \)
   \( ω_2 = 0.2 \text{ rad/sec} \)
   \( ω_3 = 2 \text{ rad/sec} \)
   \( ω_4 = 20 \text{ rad/sec} \)

### Table 6.4.

<table>
<thead>
<tr>
<th>( ω )</th>
<th>- Arg(jω)</th>
<th>- Arg(1+j 50ω)</th>
<th>+ Arg(1+j 5ω)</th>
<th>- Arg(1+j 0.5ω)</th>
<th>- Arg(1+j 0.05ω)</th>
<th>Resultant</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.1</td>
<td>90°</td>
<td>-78.69°</td>
<td>26.56°</td>
<td>-2.86°</td>
<td>-0.286°</td>
<td>-145.28°</td>
</tr>
<tr>
<td>0.2</td>
<td>90°</td>
<td>-84.29°</td>
<td>45°</td>
<td>-5.71°</td>
<td>0.573°</td>
<td>-135.57°</td>
</tr>
<tr>
<td>0.5</td>
<td>90°</td>
<td>-87.70°</td>
<td>68.2°</td>
<td>-14.04°</td>
<td>-1.43°</td>
<td>-124.57°</td>
</tr>
<tr>
<td>1.0</td>
<td>90°</td>
<td>-88.85°</td>
<td>78.69°</td>
<td>-26.56°</td>
<td>-2.86°</td>
<td>-129.38°</td>
</tr>
<tr>
<td>2.0</td>
<td>90°</td>
<td>-89.43°</td>
<td>84.29°</td>
<td>-45°</td>
<td>-5.71°</td>
<td>-145.85°</td>
</tr>
<tr>
<td>5.0</td>
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<td>-89.77°</td>
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<td>-68.2°</td>
<td>-14.04°</td>
<td>-174.8°</td>
</tr>
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<td>-71.56°</td>
<td>-16.70°</td>
<td>-179.9°</td>
</tr>
<tr>
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<td>90°</td>
<td>-89.84°</td>
<td>88.36°</td>
<td>-74.05°</td>
<td>-19.29°</td>
<td>-184.29°</td>
</tr>
<tr>
<td>10.0</td>
<td>90°</td>
<td>-89.88°</td>
<td>88.85°</td>
<td>-78.69°</td>
<td>-26.57°</td>
<td>-196.29°</td>
</tr>
</tbody>
</table>
A. Lead-Lag Compensation

In lead-lag compensation, the gain crossover frequency shifts to a higher value and therefore the phase lag compensation, speed of response becomes fast but the steady state error does not show much improvement.

In phase lag compensation, the gain crossover frequency shifts to a lower value and therefore as bandwidth decreases, speed of response reduces but the steady state error improve.

The speed of response and steady state error can be simultaneously improved if both phase lead and phase lag networks are used.

The transfer function of lead-lag network can be written as

\[
G(s) = \frac{1 + aT_1 s}{1 + T_1 s} \frac{1 + bT_2 s}{1 + T_2 s}
\]  

(6.23)

The circuit of lead-lag compensation is shown in fig. 6.5.

The equations for the network are:

\[
Z_1 = \frac{R_1}{1 + R_1 C_1 s} \quad \text{and} \quad Z_2 = \frac{1 + R_2 C_2 s}{C_2 s}
\]

(6.24)

\[
\frac{E_p(s)}{E_i(s)} = \frac{Z_2(s)}{Z_1(s) + Z_2(s)} = \frac{1 + R_2 C_2 s}{1 + R_1 C_1 s + R_2 C_2 s}
\]

(6.25)

Compare the equation 6.23 with 6.25.

\[
aT_1 = R_1 C_1
\]

(6.26)

\[
bT_2 = R_2 C_2
\]

(6.27)

\[
T_1 T_2 = R_1 R_2 C_1 C_2
\]

(6.28)

Multiplying the equation (6.26) & (6.27)

\[
adT_1 T_2 = R_1 R_2 C_1 C_2
\]

(6.29)

From eqns 6.28 and 6.29

\[
a > 1 \text{ lead compensation}
\]

\[
b < 1 \text{ lag compensation}
\]

Pole Zero Plot

\[
\begin{align*}
-1/T & \text{ lead} \\
-1/aT & \text{ lag}
\end{align*}
\]

Fig. 6.5(a)
Example 6.3. The open loop transfer function of a unity feedback is

$$G(s) = \frac{K}{s(1+0.2s)}$$

It is required that the velocity error constant should be at least 20 and the phase margin should be 44°. Does the system meet the required specifications. If not, design the compensating network to satisfy the required specifications. 

Solution: Step 1:

$$K_v = \lim_{s \to 0} sG(s)$$

$$K_v = \lim_{s \to 0} \frac{K}{s(1+0.2s)}$$

$$K_v = K = 20$$

$$G(s) = \frac{20}{s(1+0.2s)}$$

Step 2: Draw the Bode plot of the transfer function obtained in step 1. Corner frequency = 5 rad/sec.

Step 3: Phase margin available = 23°

Specified phase margin = 44°

$$\phi_m = 44° - 23° = 21°$$

$$\phi_m = 20°$$

Step 4: Calculation of 'a' and 'T':

$$\sin \phi_m = \frac{a-1}{a+1}$$

$$\sin 20° = \frac{a-1}{a+1}$$

$$a = 2.039$$

Zero frequency attenuation = -10 log a = -10 log 2.039 = -3.1 db

Draw a horizontal line at -3.1 db to cut the magnitude curve, measure the frequency at the point of intersection, this will be $$\omega_m$$.

From Bode plot $$\omega_m = 12$$ rad/sec.

Since,

$$\omega_m = \frac{1}{\sqrt{T}}$$

$$T = \frac{1}{\omega_m \sqrt{T}}$$

$$T = \frac{1}{12 \sqrt{2.039}} = 0.058$$

The transfer function of compensating network

$$G_c(s) = \frac{1}{2.039} \frac{1+0.118s}{1+0.085s}$$

The overall transfer function of the system with $$A = 2.039$$

$$G(s) = G(s). G_c(s)$$

$$G(s) = \frac{20(1+0.118s)}{s(1+0.2s)(1+0.085s)}$$

Step 5: Draw the Bode plot and investigate whether all the specifications met or not.

Corner frequencies are

$$\omega_1 = 5$$ rad/sec.

$$\omega_2 = 8.47$$ rad/sec.

$$\omega_3 = 17.24$$ rad/sec.
Example 6.6. Design a suitable phase lag compensating network for

\[ G(s) = \frac{K}{s(1+0.1s)(1+0.2s)} \]

to meet the following specifications

\[ K_p = 30 \text{ sec}^{-1} \]
\[ \text{P.M.} \geq 40^\circ \]

Solution: Step 1:

\[ K_p = \lim_{s \to 0} sG(s) = \lim_{s \to 0} \frac{K}{s(1+0.1s)(1+0.2s)} = K \]
\[ K = 30 \]

\[ G(s) = \frac{30}{s(1+0.1s)(1+0.2s)} \]

Step 2: Draw the Bode plot of the transfer function obtained in step 1:

- Corner frequencies are:
  \[ \omega_1 = 5 \text{ rad/sec.} \]
  \[ \omega_2 = 10 \text{ rad/sec.} \]

Table 6.6.

<table>
<thead>
<tr>
<th>( \omega )</th>
<th>Arg ((j\omega))</th>
<th>Arg ((1 + j0.2\omega))</th>
<th>Arg ((1 + j0.058\omega))</th>
<th>Arg ((1 + j0.118\omega))</th>
<th>Resultant</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>-90°</td>
<td>-11.3°</td>
<td>-3.31°</td>
<td>6.73°</td>
<td>-97.88°</td>
</tr>
<tr>
<td>2</td>
<td>-90°</td>
<td>-21.8°</td>
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<td>13.27°</td>
<td>-105.14°</td>
</tr>
<tr>
<td>4</td>
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<td>-38.65°</td>
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<td>25.26°</td>
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<tr>
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<td>-124°</td>
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<tr>
<td>8</td>
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<td>43.35°</td>
<td>-130°</td>
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<td>-63.4°</td>
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<td>-133.68°</td>
</tr>
<tr>
<td>20</td>
<td>-90°</td>
<td>-76°</td>
<td>-49°</td>
<td>67.03°</td>
<td>-148°</td>
</tr>
</tbody>
</table>

Step 3: From Bode Plot

- Phase margin available = -22°
- Required phase margin 40° is available at \( \omega = 3 \text{ rad/sec} \).
- Draw a line at -140° the point of intersection of this line with phase curve is 'x'. From the point 'x' draw a line on the magnitude curve.
- The gain available at \( \omega_m = 3 \text{ rad/sec} \) is 20 db (from Bode plot). \( \omega_m \) is the new cross-over frequency.
Step 4: Calculation of 'a' and 'T'

\[ |G(j\omega_0)| = 20 \log a \]
\[ -20 \log a = 20 \]
\[ a = 0.1 \]
\[ \frac{1}{aT} = \frac{\omega_n}{10} \]
\[ \frac{1}{0.1T} = \frac{2.8}{10} \quad \therefore \quad T = 35.7 \]

Note: It will be better \( \omega_n = 2.8 \text{ rad/sec} \) instead of 3 rad/sec.

Step 5: Calculation of transfer function of compensator

\[ G_c(s) = \frac{1 + aTs}{1 + Ts} \]
\[ G_c(s) = \frac{1 + 3.57s}{1 + 35.7s} \]

Step 6: Calculation of overall transfer function of the system.

\[ G(s) = \frac{30(1 + 3.57s)}{s(1 + 0.1s)(1 + 0.2s)(1 + 35.7s)} \]

Step 7: Draw the Bode plot for the transfer function obtained in step 6.

Corner frequencies are

\( \omega_1 = 0.028 \text{ rad/sec} \)
\( \omega_2 = 0.28 \text{ rad/sec} \)
\( \omega_3 = 10 \text{ rad/sec} \)
\( \omega_4 = 5 \text{ rad/sec} \)

Table 6.8.

<table>
<thead>
<tr>
<th>( \omega )</th>
<th>-Arg (( j\omega ))</th>
<th>-Arg (1 + 0.1 ( \omega ))</th>
<th>-Arg (1 + 0.2 ( \omega ))</th>
<th>-Arg (1 + 35.7 ( \omega ))</th>
<th>+Arg (1 + 3.57 ( \omega ))</th>
<th>Resultant</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.1</td>
<td>-90°</td>
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<td>+88.39°</td>
<td>-199.71°</td>
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<tr>
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<td>-87.13°</td>
<td>-89.98°</td>
<td>+89.84°</td>
<td>-261.27°</td>
</tr>
</tbody>
</table>

Example 6.5. Consider a lag lead network defined by

\[ G_c(s) = K \left( \frac{s + \frac{1}{T_1}}{s + \frac{1}{T_2}} \right) \left( \frac{s + \frac{n}{T_1}}{s + \frac{1}{\beta T_2}} \right) \]

Show that at frequency \( \omega_m \), where \( \omega_m = \frac{1}{\sqrt{T_1 T_2}} \), the phase angle of \( G_c(j\omega) \) becomes zero.
Solution: From the given transfer function put \( s = j\omega \)

\[
G_c(j\omega) = K \left( \frac{\omega_1 + \frac{\omega_1}{T_1}}{\omega_2 + \frac{\omega_2}{T_2}} \right) \left( \frac{\omega_1 + \frac{\omega_1}{\beta T_2}}{\omega_2 + \frac{\omega_2}{\beta T_2}} \right)
\]

\[
\angle G_c(j\omega) = \tan^{-1} \omega T_1 + \tan^{-1} \omega T_2 \quad \text{and} \quad \tan^{-1} \frac{T_1}{\omega T_2} - \tan^{-1} \frac{T_2}{\omega T_2} \beta
\]

Put \( \omega = \omega_1 = \frac{1}{\sqrt{T_1 T_2}} \) in equation 6.30

\[
\angle G_c(j\omega) = \tan^{-1} \frac{T_1}{\sqrt{T_2 T_1}} + \tan^{-1} \frac{T_2}{\sqrt{T_1 T_2}} - \tan^{-1} \frac{T_1}{\omega T_2} \frac{1}{\sqrt{T_1 T_2}} - \tan^{-1} \frac{T_2}{\omega T_2} \beta
\]

From equation 6.31 take first two terms

\[
\tan \left[ \tan^{-1} \frac{T_1}{\sqrt{T_2 T_1}} + \tan^{-1} \frac{T_2}{\sqrt{T_1 T_2}} \right] = \frac{T_1}{\sqrt{T_2 T_1}} + \frac{T_2}{\sqrt{T_1 T_2}} \approx \infty
\]

\[
\tan^{-1} \frac{T_1}{\sqrt{T_2 T_1}} + \tan^{-1} \frac{T_2}{\sqrt{T_1 T_2}} = \frac{\pi}{2}
\]

Similarly,

\[
\tan \left[ \tan^{-1} \frac{1}{\beta \sqrt{T_1 T_2}} + \tan^{-1} \frac{T_2}{\sqrt{T_1 T_2}} \right] = \frac{1}{\beta \sqrt{T_1 T_2}} + \frac{T_2}{\sqrt{T_1 T_2}} \approx \infty
\]

\[
\tan^{-1} \frac{1}{\beta \sqrt{T_1 T_2}} + \tan^{-1} \frac{T_2}{\sqrt{T_1 T_2}} = \frac{\pi}{2}
\]

From eq. (6.31)

\[
\angle G_c(j\omega) = \tan^{-1} \frac{T_1}{\sqrt{T_1 T_2}} + \tan^{-1} \frac{T_2}{\sqrt{T_1 T_2}} - \left( \tan^{-1} \frac{1}{\beta \sqrt{T_1 T_2}} + \tan^{-1} \frac{T_2}{\sqrt{T_1 T_2}} \right)
\]

Put the values from 6.32 & 6.33 in eq. 6.34

\[
\angle G_c(j\omega) = \frac{\pi}{2} - \frac{\pi}{2} = 0
\]

Therefore, the angle of \( G_c(j\omega) \) becomes zero at \( \omega = \frac{1}{\sqrt{T_1 T_2}} \). Proved.

Example 6.6: The open loop transfer function of a system with unity feedback is given by

\[
G(s) = \frac{K}{s^2 (1 + 0.25s)}
\]
Design a lead compensator to meet the following specifications:
1. Acceleration constant $K_a = 10$
2. P.M. = 35°

Solution: Step 1:

$$K_a = \lim_{s \to 0} s^2 C(s) = \lim_{s \to 0} \frac{s^2 - K}{s^2 (1 + 0.25s)} = K$$

$$K = K_a = 10$$

$$G(s) = \frac{10}{s^2 (1 + 0.25s)}$$

Step 2: Draw the Bode plot for uncompensated system.

<table>
<thead>
<tr>
<th>$\omega$</th>
<th>- $\text{Arg\ (j} \omega)^2$</th>
<th>- $\text{Arg\ (1 + j} \omega \cdot 0.25\omega)$</th>
<th>Resultant</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.1</td>
<td>-180°</td>
<td>-1.43°</td>
<td>-181.43°</td>
</tr>
<tr>
<td>0.2</td>
<td>-180°</td>
<td>-2.86°</td>
<td>-182.86°</td>
</tr>
<tr>
<td>0.4</td>
<td>-180°</td>
<td>-5.71°</td>
<td>-185.71°</td>
</tr>
<tr>
<td>0.8</td>
<td>-180°</td>
<td>-11.3°</td>
<td>-191.3°</td>
</tr>
<tr>
<td>1.0</td>
<td>-180°</td>
<td>-14.03°</td>
<td>-194.03°</td>
</tr>
<tr>
<td>2.0</td>
<td>-180°</td>
<td>-26.56°</td>
<td>-206.56°</td>
</tr>
<tr>
<td>3.0</td>
<td>-180°</td>
<td>-36.87°</td>
<td>-216.87°</td>
</tr>
<tr>
<td>5.0</td>
<td>-180°</td>
<td>-51.34°</td>
<td>-231.34°</td>
</tr>
<tr>
<td>10.0</td>
<td>-180°</td>
<td>-68.19°</td>
<td>-248.19°</td>
</tr>
<tr>
<td>20.0</td>
<td>-180°</td>
<td>-78.69°</td>
<td>-258.69°</td>
</tr>
<tr>
<td>30.0</td>
<td>-180°</td>
<td>-82.4°</td>
<td>-262.4°</td>
</tr>
<tr>
<td>50.0</td>
<td>-180°</td>
<td>-85.43°</td>
<td>-265.43°</td>
</tr>
</tbody>
</table>

Step 3: From Bode plot:

Specified P.M. $\phi_m = 35°$

$P.M. = 38°$

$\phi_m = 35° - (38°) + 11 = 84°$

Since $\phi_m > 60°$, two identical networks each contributing a maximum lead of $\phi_m / 2$ are used.

$\phi_m / 2 = 42°$

$\sin \phi_m = \frac{a-1}{a+1}$

$\sin 42° = \frac{a-1}{a+1}$

$\therefore a = 5.04$

Step 4: Zero freq. attenuation = $2 \left[ -10 \log a \right]$

$= -14.04 \, \text{db}$

Draw a line at $-14.04 \, \text{db}$ on magnitude curve. From graph $\omega_m = 6 \, \text{rad/sec}$.

Step 5: Calculation of $T$

$$\omega_m = \frac{1}{\sqrt{a \cdot T}}$$

Table 6.10.

<table>
<thead>
<tr>
<th>$\omega$</th>
<th>-$\text{Arg(j}\omega)^2$</th>
<th>-$\text{Arg(1+j}0.25\omega)^2$</th>
<th>-$\text{Arg(1+0.374}\omega)^2$</th>
<th>Resultant</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.1</td>
<td>-180°</td>
<td>-1.43°</td>
<td>-0.85°</td>
<td>-178°</td>
</tr>
<tr>
<td>0.2</td>
<td>-180°</td>
<td>-2.86°</td>
<td>-1.69°</td>
<td>-176°</td>
</tr>
<tr>
<td>0.4</td>
<td>-180°</td>
<td>-5.71°</td>
<td>-3.39°</td>
<td>-172.08°</td>
</tr>
<tr>
<td>0.8</td>
<td>-180°</td>
<td>-11.3°</td>
<td>-6.76°</td>
<td>-164.75°</td>
</tr>
<tr>
<td>1.0</td>
<td>-180°</td>
<td>-14.03°</td>
<td>-8.46°</td>
<td>-161.48°</td>
</tr>
<tr>
<td>2.0</td>
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<td>-26.56°</td>
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<td>-149.8°</td>
</tr>
<tr>
<td>3.0</td>
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<td>-36.87°</td>
<td>-25.0°</td>
<td>-145.29°</td>
</tr>
<tr>
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<td>-51.34°</td>
<td>-40.6°</td>
<td>-148.21°</td>
</tr>
<tr>
<td>10.0</td>
<td>-180°</td>
<td>-68.19°</td>
<td>-73°</td>
<td>-171.19°</td>
</tr>
<tr>
<td>20.0</td>
<td>-180°</td>
<td>-78.69°</td>
<td>-111.9°</td>
<td>-205.82°</td>
</tr>
<tr>
<td>30.0</td>
<td>-180°</td>
<td>-82.4°</td>
<td>-131.5°</td>
<td>-224°</td>
</tr>
<tr>
<td>50.0</td>
<td>-180°</td>
<td>-85.43°</td>
<td>-149.75°</td>
<td>-241.31°</td>
</tr>
</tbody>
</table>

5.4 Compensator using Root Locus

Root locus method can also be used to design a compensator. When an improvement is required in steady state performance, a lag compensator is designed. The poles and zeros of lag compensator are place in such a way that it does not affect the root loci carrying the dominant closed loop poles. First plot the root locus for uncompensated system and determine the value of $K$ for desired damping ratio and obtain the dominant poles. If steady state specifications are not met design a lag compensator. The pole and zero of the compensator should be closed to the origin so that root locus remains the same.

When an improvement in transient performance is required lead compensator should be designed. The poles and zeros of the lead compensator should be so placed that the resulting closed loop transfer function has a pair of complex poles in the left half of s-plane with the required damping ratio and natural frequency $\omega_n$, and also all other poles are either near a zero or much
A.4.3 Lag Compensation

The compensator having the transfer function

\[ G_c(s) = \frac{1 + aT_s}{1 + Ts} \quad a < 1 \]

is known as lag compensator. The lag compensator improves the steady state performance of the system.

Example 6.7. Design a lag compensator for a system whose open loop transfer function is

\[ G(s) = \frac{K}{s(s+1)(s+4)} \]

to meet the following specifications.

- Damping ratio \( \zeta = 0.5 \)
- Setting time \( t_s = 10 \) sec.
- Velocity error constant \( K_v \geq 5 \)

Solution: Step 1: Draw the root locus for uncompensated system.

1. Number of Poles \( P = 3 \)
   - \( s_1 = 0, s_2 = -1, s_3 = -4 \)
   - \( z = 0 \)

Plot the pole and zero

2. The segment between \( s = 0 \) and \( s = -1 \), from \( s = -4 \) to onwards are the parts of the root locus.

3. Centroid of the asymptotes
   \[ \sigma_z = \frac{\text{sum of poles} - \text{sum of zeros}}{P - Z} = \frac{-1 - 4}{3} = -5/3 \]

4. Angles of asymptotes
   - \( \phi = \frac{2K + 1}{P - Z} \times 180^\circ \)
   - \( K = 0 \) \( \phi = 60^\circ \)
   - \( K = 1 \) \( \phi = 180^\circ \)
   - \( K = 2 \) \( \phi = 300^\circ \)

5. Breakaway point

The characteristic equation \( 1 + G(s)H(s) = 0 \)

- \( \alpha_1 (s+1)(s+4) + K = 0 \)
- \( \alpha_2 s^3 + 5 s^2 + 4 s + K = 0 \)
- \( \alpha_3 \)

\[ K = -(3 s^3 + 10 s + 4) = 0 \]

\[ 3 s^3 + 10 s + 4 = 0 \]

\[ s = -2.86, -0.46 \]
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\( s = 0.46 \) is the breakaway point

(\( \ell \)) point of intersection of root locus with \( j\omega \) axis
characteristic \( eq'' \). \( s^3 + 5s^2 + 4s + K = 0 \)

Routh array

<table>
<thead>
<tr>
<th>( s )</th>
<th>1</th>
<th>4</th>
</tr>
</thead>
<tbody>
<tr>
<td>( s^2 )</td>
<td>5</td>
<td>( K )</td>
</tr>
<tr>
<td>( s^3 )</td>
<td>( 20 - K )</td>
<td>( \frac{5}{\ell} )</td>
</tr>
<tr>
<td>( s^4 )</td>
<td>( \ell )</td>
<td></td>
</tr>
</tbody>
</table>

When \( K = 20 \), the auxiliary equation

\[ A(s) = 5s^2 + K = 0 \]
\[ 5s^2 + 20 = 0 \]
\[ 5s^2 = -20 \]
\[ s^2 = -4 \]
\[ s = \pm 2i \]

The complete root locus is shown in the fig 6.9

Step 2: Draw a line for \( \zeta = 0.5 \)
\[ \cos^{-1} 0.5 = 60^\circ \]
Draw a line at angle of 60\(^\circ\) with negative real axis.

Step 3: Find the value of \( K \) at the point \( A \)

\[ OA = 0.8 \]
\[ AB = 0.9 \]
\[ AC = 3.7 \]
\[ K = OA \times AB \times AC = 0.8 \times 0.9 \times 3.7 = 2.664 \]

Step 4: The velocity constant for the cascade system should satisfy

\[ K_v = \lim_{s \to 0} s \left( \frac{s + 1}{s + \frac{1}{T}} \right) \frac{2.664}{s(s+1)(s+4)} \]
\[ 5 = \frac{2.664}{a} \]
\[ a = 0.13 \]

The lag compensator should be so placed that the root locus is not disturbed by putting in the compensator pole and zero near the origin. Generally, 10% of distance between the origin and first pole on the negative real axis is taken for the location of compensator zero

\( a = \frac{1}{T} \) is at -0.1 (10% of \( 1 \))
\( -\frac{1}{T} = 0.1 \)
\( a = 0.13 \)
\( \frac{1}{T} = 0.013 \)

Hence, the transfer function of compensator \( G_c = \frac{s + 0.1}{s + 0.013} \)

The overall transfer function = \( G(s) \) \( \frac{G(s)}{G(s)} \)

\[ G(s) = \frac{2.664(s + 0.1)}{(s + 1)(s + 4)(s + 0.013)} \]

Example 6.8. For example (5.23) design a lag compensator for

\[ K_0 \geq 5 \]
\[ \zeta = 0.707 \]
\[ \omega_n = 2 \text{ rad/sec}. \]

Solution: The complete root locus is shown in fig 6.10. (For calculations readers please see the example 5.23)

The value of \( K \) at the point \( A \):

\[ K = OA \times AB \times AC = 1.8 \times 3.2 \times 4.1 = 23.6 \]

The velocity constant for cascade system should satisfy

\[ K_v = \lim_{s \to 0} s \left( \frac{s + 1}{s + \frac{1}{T}} \right) \frac{23.6}{s(s+4)(s+5)} \]
\[ 5 = \frac{23.6}{a} \]
\[ a = 0.236 \]

Take the distance 10% of the first pole from the origin for compensator zero i.e. \( \frac{1}{T} \) will be placed at -0.4 from origin.

\[ \frac{1}{T} = 0.4 \]
\[ \therefore \frac{1}{T} = 0.0944 \]

The transfer function of the compensator \( G_c = \frac{s + 0.4}{s + 0.0944} \)

The overall transfer function after compensation \( G'(s) = \frac{23.6(s + 0.4)}{(s + 4)(s + 5)(s + 0.0944)} \)

4.2 Lead Compensation

The following steps are involved for designing a lead compensator:

1. Draw the root locus for the given open-loop transfer function.
2. From the transient response specification determine the desired location for the dominant closed-loop poles by

\[ \text{Eq} = \frac{-\xi \omega_n}{\sqrt{1-\xi^2}} \]

Where, \( \xi \) is the damping ratio and \( \omega_n \) is the undamped natural frequency. The damping ratio determines the angular location of the poles and distance of the pole from origin is determined by undamped natural frequency \( \omega_n \).
3. Connect all the open-loop poles and zeros to one of the dominant closed loop poles (at desired location) and measure the angles & then find the sum of angles. Now calculate the angle deficiency \( \phi \) to be added, so that sum of angle is equal to \( 180(2k+1) \) at that point.
4. If the original system has the open-loop transfer function \( G(s) \), then the compensated system will have open-loop transfer function.

\[
G_c(s) G(s) = \left( k_c \frac{s + \frac{1}{s}}{s + \frac{1}{\alpha T}} \right) G(s)
\]

where,

\[
G_c(s) = k_c \frac{1 + sT}{1 + \alpha s}
\]

\( \alpha \) and \( T \) can be determined from angle deficiency. \( k_c \) is determined from the requirement of the open-loop gain.

5. Determine the open-loop gain of the compensated system from the magnitude condition.

6. Now check whether all performance specifications have been met or not.

Example 6.9. For the system shown above, design a lead compensator such that \( \omega_n = 4 \text{ rad/sec} \) and \( \xi = 0.5 \) for compensated system.

![Diagram of control system](image)

**Fig. 6.11.**

**Solution:** Open-loop transfer function

\[
G(s) H(s) = \frac{4}{s(s + 2)}
\]

**Step 1:** Draw the root locus of given open-loop transfer function.

**Note:** Students are advised to draw the root-locus by themselves.

**Step 2:** Plot the dominant closed-loop poles at desired location. Here, \( \xi = 0.5 \) and \( \omega_n = 4 \text{ rad/sec} \) given

\[
s_d = \frac{\xi \omega_n \pm j \omega_n \sqrt{1 - \xi^2}}{2}
\]

\[
s_d = -2 \pm j3.46
\]

**Step 3:** Connect all the poles to point \( P \) and then find the angle deficiency.

\[
\left. \frac{4}{s(s + 2)} \right|_{s = -2 \pm j3.46} = -210^\circ
\]

[Alternative: \( 180 - (90 + 120) = 30^\circ \)]

Thus lead compensator must contribute \( \phi = 30^\circ \) at this point.

**Step 4:** Determine location of zeros and pole of the lead compensator as follows.

Draw a horizontal line passing through point \( P \) to \( A \) connect the point \( P \) to the origin. Measure the angle \( \angle APO \) and bisect it. \( \phi \) is the bisector. Draw two lines \( PC \) & \( PD \) from \( P \). A line that makes angle \( \pm \phi /2 = 30/2 = 15^\circ \). The intersection of \( PC \) & \( PD \) with negative real axis gives the location for zero and pole of lead network.
Step 5: From the plot
zero at $s = -2.96$
Pole at $s = -5.5$

\[ \frac{1}{T} = 2.96 \quad \text{&} \quad \frac{1}{\alpha T} = 5.5 \]

\[ T = 0.337 \quad \alpha = 0.538 \]

Therefore open-loop transfer function of compensated system is

\[ G_c(s) G(s) = k_c \frac{s + 2.96}{s + 5.5} \frac{4}{s(s + 2)} = \frac{k_c(s + 2.96)}{s(s + 2)(s + 5.5)} \]

Where

\[ k_c = 4K_e \]

Step 6: Calculate $K_c$ from magnitude condition

\[ \left| \frac{k_c(s + 2.96)}{s(s + 2)(s + 5.5)} \right|_{m = -2, \pm 3.46} = 1 \]

\[ K_c = 18.7 \]

\[ K_c = 4K_e \]

\[ K_e = \frac{18.7}{4} = 4.675 \]

\[ k_c = 4.675 \times 0.538 = 2.52 \]

Step 7: Calculate $K_c$

\[ K_c = \frac{k_c}{4} = \frac{18.7}{4} = 4.675 \]

\[ K_c \alpha = 4.675 \times 0.538 = 2.52 \]

\[ \text{Transfer function of lead compensator} = 2.52 \frac{1 + 0.337s}{1 + 0.182s} \]

or,

\[ G_c(s) = \frac{4.675(s + 2.96)}{s(s + 5.5)} \]

Also, open loop transfer function of compensated system

\[ G_c(s) G(s) = \frac{18.7(s + 2.96)}{s(s + 2)(s + 5.5)} \]

(put the value of $K_c$ in equation 6.40)

Step 8: We can calculate velocity error constant $K_v$

\[ K_v = \lim_{s \to 0} s G_c(s) G(s) = \lim_{s \to 0} s \frac{18.7(s + 2.9)}{s(s + 2)(s + 5.5)} \]

\[ K_v = 5.02 \text{ sec}^{-1} \]

Summary

Compensation technique is used to make the unstable system stable by introducing the poles and/or zeros at suitable places.

There are many types of compensation depending upon the location of the compensator network. Such as series compensation, feedback compensation and lead compensation.

Transfer function of phase lead network is given by

\[ \frac{E_2(s)}{E_1(s)} = \frac{1}{a} \left[ \frac{1 + aTs}{1 + Ts} \right] \]
Chapter 7

Non-Linear System Analysis

7.1. INTRODUCTION

In a linear system the form of output does not depend on the magnitude of the input, when the input increases the output will also increases but the form remains the same. In non-linear systems form may change with change in magnitude of the input. For example in a linear system, if the input is sinusoidal the output of the linear system will also be sinusoidal but in non linear the output will be non-sinusoidal. The non linear systems do not possess the homogeneity and superposition properties. The stability of the non-linear systems depends on the input and the initial state of the system. If a non-linear system, the input and outputs are described by the non linear differential equation then it is called dynamic non-linearity and if the input and output are not described by the differential equations then it is known as static non linearity. Sometime to improve the characteristic of the system some non-linearities are intentionally added to the system. Such type of non-linearities are called intentional non-linearities. If the non-linearities are inherently present in the system (e.g. saturation, dead zone, friction etc.) then these non-linearities are known as incidental non-linearities.

7.2 SOME COMMON TYPES OF NON-LINEARITIES

Some basic common non-linearities are discussed in brief.

7.2.1. Saturation

The saturation characteristic is shown in fig. 7.1. The common example of saturation is the relationship between magnetic flux and current in an iron cored coil. An iron core coil has linear inductance for low values of current but if the current increases, the core becomes linear and inductance becomes non-linear. Another example is a resistor, a resistor is linear for low values of current but becomes non-linear for higher values of current because of the temperature. Transistors, operational amplifiers and magnetic components have this type of non-linearity.

*If the magnitude of the input increase 'K' times, the output magnitude also increased 'K' times then this property is called homogeneity. e.g. if the input is 'x' and output is 'y' and the input becomes Kx then the output magnitude should be Ky.

If the two inputs are applied simultaneously, then output will be the sum of two outputs. This property is called superposition. e.g., if one input is x₁ and output y₁ and other input is x₂ and output y₂ then x₁ + x₂ → y₁ + y₂.

7.2.2. Friction

Frictional force opposes the motion of a body. The friction is of types (a) static friction (b) dynamic friction. Static friction is the friction when the body tends to move and the dynamic friction is experienced by a body when in motion. The dynamic friction can further be classified as sliding friction and rolling friction. The sliding friction experienced by a body when it slides over another body. The rolling friction experienced by a body when it rolls over the other. In mechanical system there are two types of friction (i) viscous friction (ii) static friction. Viscous friction is proportional to the relative velocity between the moving surfaces. The force-velocity relationship is linear. This characteristic is shown in fig. 7.3.

7.2.3. Relays

A relay is a non-linear power amplifier which provides a large power amplification, therefore they are used in control system. The characteristic of ideal relay is shown in fig. 7.4a. The actual relay having a dead zone as shown in fig. 7.4b.

7.2.4. Backlash

Backlash is the play between the teeth of the drive gear and driven gear. The characteristic is shown in fig. 7.5.

7.3. STUDY OF NON-LINEAR SYSTEMS

For the study of non-linear systems we shall consider the following methods:

![Figures and diagrams related to non-linear system analysis are shown.]
7.3.1. Describing Function Method
Consider the fig. 7.6. A non-linear device is represented by a block. If we apply a sine wave at amplitude $A$ and frequency $\omega$ then the actual output of a non-linear system does not have sinusoidal wave shape but it is periodic. The Fourier series of non-sinusoidal wave form contains the

If the input is given by $A \sin \omega t$, then the steady state output may be expressed in the following Fourier series

$$y(t) = \frac{A}{2} + A_1 \cos \omega t + B_1 \sin \omega t + A_2 \cos 2\omega t + B_2 \sin 2\omega t + ...$$  \hspace{1cm} (7.1)

The first two components of equation (7.1) are the fundamental components and others are harmonics. For writing the equation (7.1) it is assumed that non-linearity does not generate subharmonics and if non-linearity is symmetrical then the average value of $y(t)$ will be zero so the output is given by

$$y(t) = A_1 \cos \omega t + B_1 \sin \omega t + A_2 \cos 2\omega t + B_2 \sin 2\omega t + ...$$  \hspace{1cm} (7.2)

![Fig. 7.6.](image)

Fig. 7.6.

The describing function is defined as the ratio of amplitudes and phase angle between the fundamental components of the output and the input sinusoidal, for all amplitudes of the input which are to be considered and for all frequencies from zero to $+\infty$.

$$G_D(j\omega) = \frac{\text{Magnitude and phase angle of output}}{\text{Magnitude and phase angle of input}}$$

7.3.2. Limitations
(a) If the system having one non-linear part can be analysed by this method. If the system having two non-linear part, this method is not suitable because the product of two describing function $G_D'(j\omega)$ and $G_D''(j\omega)$ is defined.
(b) Non-linear devices which produces periodic waves having large amplitude harmonic present more filtration problem.
(c) The location of non-linear device may affect the stability.

7.3.3. Describing Function for Saturation
This generally occurs in amplifiers. The saturation characteristic is shown in fig. 7.7. The waveform of the output is shown in fig. 7.8.

![Fig. 7.7.](image)

![Fig. 7.8.](image)

$E_i$ = Value of input required to saturate the system.

Let input

$$e_i = E \sin \omega t$$

output

$$e_o = K e_i = 0 \leq \omega t \leq \beta$$

The output will remain constant from $\beta$ to $(\pi - \beta)$

$$e_o = K E_s \beta \leq \omega t \leq (\pi - \beta)$$

$$e_o = K E_s \pi - \beta \leq \omega t \leq \pi$$

Calculation of $A_1$:

$$A_1 = \frac{2}{\pi} \int_0^\pi y(t) \cos \omega t \, dt$$

$$A_1 = \frac{2}{\pi} \left[ K E_s \sin \omega t \cos \omega t \int_0^\beta \sin \omega t \cos \omega t \, dt + \frac{\pi}{2} K E_s \cos \omega t \int_0^\beta \cos \omega t \sin \omega t \, dt + \frac{\pi}{2} K E_s \right]$$

$$A_1 = \frac{2}{\pi} \left[ K E_s \frac{\pi}{2} \cos \omega t \int_0^\beta \sin 2\omega t \, dt + \frac{\pi}{2} \cos \omega t \int_0^\beta \cos 2\omega t \, dt + \frac{\pi}{2} K E_s \cos \omega t \int_0^\beta \sin 2\omega t \, dt \right]$$

$$A_1 = 0$$
CALCULATION OF $B_1$:

$$B_1 = \frac{2}{\pi} \int_0^\beta y(t) \sin \omega t \, d\omega$$

$$= \frac{2}{\pi} \left[ KE \sin \omega t \sin \omega t \, d\omega + \frac{1}{2} K E \sin \omega t \sin \omega t \, d\omega \right]$$

$$= \frac{2}{\pi} \left[ KE \sin^2 \omega t \, d\omega \right] + KE \sin \omega t \sin \omega t \, d\omega$$

$$= \frac{2}{\pi} \left[ KE \frac{1 - \cos 2\omega t}{2} \, d\omega \right]$$

$$= \frac{2}{\pi} \left[ KE \left( \beta - \frac{1}{2} \sin 2\omega t \right) \right]$$

$$= \frac{2}{\pi} \left[ KEB + 2KE \cos \beta - KE \sin \beta \cos \beta \right]$$

$$= \frac{2}{\pi} \left[ KEB + 2KE \cos \beta \right]$$

Since,

$$e_0 = E \sin \omega t$$

Put $e_0 = E \sin t$ and $\omega t = \beta$

$$E_0 = E \sin \beta$$

$$\sin \beta = \frac{E_0}{E}$$

$$\cos \beta = \sqrt{1 - \left(\frac{E_0}{E}\right)^2}$$

$$B_1 = \frac{2KE}{\pi} \sin^{-1} \frac{E_0}{E} + \frac{E_0}{E} \sqrt{1 - \left(\frac{E_0}{E}\right)^2}$$

$$G_D(\omega) = \frac{\sqrt{A_1^2 + R^2}}{E} \tan^{-1} \frac{A_1}{B_1}$$

Put the values of $A_1$ and $B_1$.

Required describing function

$$G_D(\omega) = \frac{1}{E} \frac{2KE}{\pi} \sin^{-1} \frac{E_0}{E} + \frac{E_0}{E} \sqrt{1 - \left(\frac{E_0}{E}\right)^2}$$

$$= \frac{2KE}{\pi} \sin^{-1} \frac{E_0}{E} + \frac{E_0}{E} \sqrt{1 - \left(\frac{E_0}{E}\right)^2}$$

If $E_0 = R$, then

$$G_D(\omega) = \frac{2KE}{\pi} \sin^{-1} R + R \sqrt{1 - R^2} \angle 0^\circ$$

13.4. Describing Function for Ideal Relay

**Diagram**

Response of the ideal relay

Let input to the relay $x = X \sin \omega t$

output

$$e_0 = Y \quad 0 \leq \omega t \leq \pi$$

$$= -Y \quad \pi \leq \omega t \leq 2\pi$$

$$A_1 = \frac{2}{\pi} \int_0^\pi \cos \omega t \, d\omega t$$

$$= \frac{2}{\pi} \int_0^\pi \cos \omega t \, d\omega t$$

$$= \frac{2}{\pi} \left[ \frac{1}{2} \sin \omega t \right]_0^\pi$$

$$= 0$$

$$A_1 = 0$$

*The output in fig 7.8 has odd symmetry, its Fourier series will have only sine terms.*
**Relay with Dead Zone (Practical Relay)**

Let the input \( x = X \sin \omega t \)

\[
\begin{align*}
\theta_0 &= 0, \quad 0 \leq \omega t \leq \alpha \\
\theta &= Y, \quad \alpha \leq \omega t \leq (\pi - \alpha) \\
\theta &= 0, \quad (\pi - \alpha) \leq \omega t \leq (\pi + \alpha) \\
\theta &= -Y, \quad (\pi + \alpha) \leq \omega t \leq (2\pi - \alpha) \\
\theta &= 0, \quad (2\pi - \alpha) \leq \omega t \leq 2\pi
\end{align*}
\]

\[
\begin{align*}
A_1 &= \frac{2}{\pi} Y \alpha \left[ \sin \omega t \right]_{\alpha}^{\pi - \alpha} \\
A_1 &= 0
\end{align*}
\]

\[
B_1 = \frac{2}{\pi} Y \left[ \sin \omega t \right]_{\alpha}^{\pi - \alpha} \cos \omega t = \frac{4Y}{\pi} \cos \alpha
\]

where

\[
\begin{align*}
\omega t &= \alpha \\
x &= d/2 = X \sin \alpha \\
\sin \alpha &= \frac{d}{2X} \\
\cos \alpha &= \left[ 1 - \left( \frac{d}{2X} \right)^2 \right]^{1/2} \\
B_1 &= \frac{4Y}{\pi} \left[ 1 - \left( \frac{d^2}{4X^2} \right) \right]^{1/2}
\end{align*}
\]
\[ A_1 = \frac{2K}{\pi} \left[ \frac{s}{2} \right] \sin 2\omega t \cos \omega t dt \cdot \cos \omega t \sin \omega t + \left( s - \frac{D}{2} \right) \cos \omega t \sin \omega t \] 

\[ = 0 \]

\[ A_1 = 0 \]

\[ B_1 = \frac{2K}{\pi} \int y(t) \sin \omega t dt \]

\[ B_1 = \frac{2K}{\pi} \int K \left( X \sin \omega t - \frac{D}{2} \right) \sin \omega t dt + \frac{\pi}{\beta} \int K \left( s - \frac{D}{2} \right) \sin \omega t dt + \frac{\pi}{\beta} \int K \left( X \sin \omega t - \frac{D}{2} \right) \sin \omega t dt \]

\[ \int \sin 2\omega t \cos \omega t dt = \frac{1}{\omega} \sin 2\omega t \sin \omega t \]

\[ \int \sin 2\omega t \sin \omega t dt = \frac{1}{\omega} \sin 2\omega t \sin \omega t \]

\[ \int \cos 2\omega t \sin \omega t dt = \frac{1}{2} \cos 2\omega t \sin \omega t \]

\[ \int \cos 2\omega t \cos \omega t dt = \frac{1}{2} \cos 2\omega t \cos \omega t \]

\[ B_1 = \frac{2K}{\pi} \left[ X \beta - X \alpha - \frac{X}{2} \sin 2\beta + \frac{X}{2} \sin 2\alpha + 2s \cos \beta - D \cos \alpha \right] \]  

Input 
\[ x = X \sin t \]

\[ \omega t = \alpha, \frac{D}{2} = X \sin \alpha \] or \( D = 2X \sin \alpha \)

\[ \sin \alpha = \frac{D}{2X} \]

\[ b = \beta, s = X \sin \beta \]

\[ \sin \beta = \frac{s}{X} \]

Put these values in eq (7.9)

\[ B_1 = \frac{2K}{\pi} \left[ X \beta - X \alpha - \frac{X}{2} \sin 2\beta + \frac{X}{2} \sin 2\alpha + 2X \sin \beta \cos \beta - 2X \sin \alpha \cos \alpha \right] \]

\[ = \frac{2K}{\pi} \left[ X(\beta - \alpha) - \frac{1}{2} X \sin 2\beta + \frac{1}{2} X \sin 2\alpha + X \sin 2\beta - X \sin 2\alpha \right] \]

\[ = \frac{2K}{\pi} \left[ (\beta - \alpha) + \frac{1}{2} \sin 2\beta - \frac{1}{2} \sin 2\alpha \right] = \frac{2K}{\pi} \left[ 2(\beta - \alpha) + \sin 2\beta - \sin 2\alpha \right] \]

\[ = \frac{KX}{\pi} \left[ 2(\beta - \alpha) + \sin 2\beta - \sin 2\alpha \right] \]

9.6 Describing Function of Backlash

\[ G_0(\infty) = \frac{A_1^2 + B_1^2}{X} \tan^{-1} A_1/B_1 = \frac{B_1}{X} \tan^{-1} \theta' \]

\[ G_0(\omega) = \frac{X}{\pi} \left( 2(\beta - \alpha) + \sin 2\beta - \sin 2\alpha \right) \]

The describing function is given by

\[ G_0(\omega) = \begin{cases} 
\frac{1}{\pi} \frac{2(\beta - \alpha) + \sin 2\beta - \sin 2\alpha}{X} & \text{if } 0 < X < D/2, \alpha = \beta = \pi/2 \\
\frac{K}{\pi} \left[ 2(\beta - \alpha) + \sin 2\beta - \sin 2\alpha \right] & \text{if } D/2 < X < s \\
\frac{K}{\pi} \left[ X \sin \omega t - \frac{D}{2} \right] & \text{if } X > s 
\end{cases} \]

Fig. 7.12
\[ A_1 = \frac{2}{\pi} \int_0^{\pi/2} y(t) \cos \omega t \, dt \]
\[ A_1 = \frac{2K}{\pi} \left[ \int_0^{\pi/2} \left( I \sin \omega t - \frac{b}{2} \right) \cos \omega t \, dt + \int_{\pi/2}^{\pi} \left( I - \frac{b}{2} \right) \cos \omega t \, dt \right] 
+ \int_{\pi/2}^{\pi} \left( I \sin \omega t + \frac{b}{2} \right) \cos \omega t \, dt \]
\[ A_1 = \frac{2K}{\pi} \left[ \int_0^{\pi/2} I \sin \omega t \cos \omega t \, dt - \frac{b}{2} \int_0^{\pi/2} \cos \omega t \, dt + \left( I - \frac{b}{2} \right) \int_{\pi/2}^{\pi} \cos \omega t \, dt \right] 
+ \int_{\pi/2}^{\pi} I \sin \omega t \cos \omega t \, dt + \frac{b}{2} \int_{\pi/2}^{\pi} \cos \omega t \, dt \]
\[ \sin \beta = \frac{I - b}{I} \]
\[ A_1 = \frac{2K}{\pi} \left[ \frac{3}{4} I + \sin \beta - b \sin \beta + \frac{I}{4} \cos 2\beta \right] \]
\[ A_1 = \frac{2K}{\pi} \left[ \frac{3}{4} I + \sin \beta (I - b) + \frac{I}{4} \left( 1 - 2 \sin^2 \beta \right) \right] \]
\[ A_1 = \frac{2K}{\pi} \left[ \frac{3}{4} I + \frac{b}{I} (I - b) + \frac{I}{4} \left( 1 - 2 \left( \frac{I - b}{I} \right)^2 \right) \right] = \frac{2K}{\pi} \left[ \frac{b^2 - 4lb}{4l^2} \right] \]

Multiplying and divide by \( I \)
\[ A_1 = \frac{4KI}{\pi} \left[ \frac{b^2 - 2lb}{4l^2} \right] = \frac{4KI}{\pi} \left[ \frac{(b/2)^2}{l^2} - \frac{b/2}{l} \right] \]
\[ A_1 = \frac{4KI}{\pi} \left[ \frac{(b/2)^2}{l^2} - \frac{(b/2)}{l} \right] \]

\[ B_1 = \frac{2}{\pi} \int_0^{\pi/2} y(t) \sin \omega t \, dt \]
\[ B_1 = \frac{2}{\pi} \int_0^{\pi/2} K \left( I \sin \omega t - \frac{b}{2} \right) \sin \omega t \, dt + \int_{\pi/2}^{\pi} K \left( I - \frac{b}{2} \right) \sin \omega t \, dt \]
\[ B_1 = \frac{2K}{\pi} \left[ \int_0^{\pi/2} I \sin^2 \omega t \, dt - \frac{b}{2} \int_0^{\pi/2} \sin \omega t \, dt + \left( I - \frac{b}{2} \right) \int_{\pi/2}^{\pi} \sin \omega t \, dt \right] 
+ \int_{\pi/2}^{\pi} I \sin^2 \omega t \, dt + \frac{b}{2} \int_{\pi/2}^{\pi} \sin \omega t \, dt \]

where
\[ I = \text{input (maximum)} \]
\[ \frac{b}{2} = \text{half backlash} \]

\[ A_1 = \frac{4KI}{\pi} \left[ R^2 - R \right] \]
\[ B_1 = \frac{K}{\pi} \left[ \frac{\pi}{2} + \sin^{-1}(1 - 2R) + \frac{b}{I} \frac{2R}{b - 1} \right] \]

Put the values of \( A_1 \) and \( B_1 \) in equation (7.11) and simplify
\[ G_0(j\omega) = \frac{K}{\pi} \left[ \frac{\pi}{2} + \sin^{-1}(1 - 2R) \right] + 4R \left( 1 - R \right) + \left( \pi + 2 \sin^{-1} \left( 1 - 2R \right) \right) \frac{1}{2R(1 - 2R) - \sqrt{\frac{1}{R} - \frac{1}{R}}} \]

\[ G_0(j\omega) = \tan^{-1} \left( \frac{4R(1 - R)}{\pi/2 + \sin^{-1}(1 - 2R)} + 2R(1 - 2R) \sqrt{\frac{1}{R} - \frac{1}{R}} \right) \]
Example 7.1. Determine the describing function of the following non-linearity.

\[ y(t) = M + Kx \sin at \]

\[ y(t) = A_x + \sum_{n=1}^{\infty} (A_n \cos nat + B_n \sin nat) \]

For symmetrical non-linearity \( A_x = 0 \)

\[ y(t) = M + Kx \sin at \]

Since, \( y(t) \) is an odd function, its Fourier series has only sine terms.

Fundamental harmonic components is

\[ y(t) = B_1 \sin at \]

\[ B_1 = \frac{1}{\pi} \int_{0}^{\pi} y(t) \sin at \, dt = \frac{4}{\pi} \int_{0}^{\pi/2} (M + Kx \sin at) \sin at \, dt \]

\[ = \frac{4}{\pi} \int_{0}^{\pi/2} (M \sin at + Kx \sin^2 at) \, dt = \frac{4}{\pi} \left[ \frac{M}{2} \sin at + \frac{Kx}{2} (1 - \cos 2at) \right]_{0}^{\pi/2} \]

\[ = \frac{4M}{\pi} + Kx \]

\[ G_D(f) = \frac{A_1^2 + B_1^2}{x} \]

\[ G_D(f) = \frac{4M}{\pi} + K \]

Ans.

Example 7.2. Determine the describing function of the following non-linearity.

\[ \sin at_1 = \frac{\Delta}{X} \]

\[ \cos at_1 = \sqrt{1 - \left(\frac{\Delta}{X}\right)^2} \]
Example 7.3. Derive the expression for describing function of the following non-linearity.

Solution:

\[ y(t) = K_1 X \sin(\omega t) \]

\[ 0 < t < t_1 \]

\[ y(t) = K_1 X \sin(\omega t) + (K_1 - K_2) \sin\left(\frac{s}{X}t\right) \]

\[ t_1 < \frac{\pi}{\omega} - t_1 \]

\[ y(t) = K_1 X \sin(\omega t) \]

\[ \frac{\pi}{\omega} - t_1 < t < \frac{\pi}{\omega} \]

7.4 Stability Analysis with Describing Function

Consider the fig. 7.16, where \( G_1(j\omega), G_2(j\omega) \) are the linear part of the system and non-linearity is replaced by the describing function \( G_D(j\omega) \). For above fig 7.16.
The characteristic equation is
\[ \frac{C(j\omega)}{R(j\omega)} = \frac{G_1(j\omega)G_D(X,j\omega)G_2(j\omega)}{1 + G_1(j\omega)G_D(X,j\omega)G_2(j\omega)} \]

According to Nyquist stability criterion, the system will exhibit sustained oscillations or limit cycle* when
\[ G_D(X,j\omega)G_2(j\omega) = -1 \]
This implies that the plot of \( G_1(j\omega)G_D(X,j\omega)G_2(j\omega) \) passes through the critical point. Generally
\[ G_1(j\omega)G_2(j\omega) = -\frac{1}{G_D(X,j\omega)} \]

Let \( G_1(j\omega)G_2(j\omega) \) plot be superimposed on \( -\frac{1}{G_D(X,j\omega)} \). When \( G_D(j\omega) \) is a function of amplitude only, the resulting plot is a single curve but if it is a function of frequency and amplitude then there is a family of curves. Then three situations arises.

(a) When there is no intersection between \( G_1(j\omega)G_2(j\omega) \) and \( -\frac{1}{G_D(X,j\omega)} \) and the plot of \( -\frac{1}{G_D(X,j\omega)} \) is completely enclosed by \( G_1(j\omega)G_2(j\omega) \) as shown in fig. 7.17a, then the system is absolutely unstable.

(b) When there is no intersection between \( G_1(j\omega)G_2(j\omega) \) and \( -\frac{1}{G_D(X,j\omega)} \) and the plot of \( -\frac{1}{G_D(X,j\omega)} \) does not enclose by the plot of \( G_1(j\omega)G_2(j\omega) \) as shown in fig. 7.17b then the system is absolutely stable.

(c) When there is no intersection between \( G_1(j\omega)G_2(j\omega) \) and \( -\frac{1}{G_D(X,j\omega)} \) and few curves are enclosed and few curves of \( -\frac{1}{G_D(X,j\omega)} \) does not enclose by \( G_1(j\omega)G_2(j\omega) \) as shown in fig. 7.17c then the system is unstable for \( \alpha_1, \alpha_2 \) and stable for \( \alpha_3, \alpha_4 \).

Now consider another case i.e. when there is a point of intersection of \( G_1(j\omega)G_2(j\omega) \) with \( -\frac{1}{G_D(X,j\omega)} \) (as shown in fig. 7.18a).

Suppose a system is actuated by a small disturbance, so a signal of small amplitude is obtained so that the point on \( -\frac{1}{G_D(X,j\omega)} \) locus corresponding to this small signal will be inside the \( G_D(j\omega) \) \( G_2(j\omega) \) locus. Hence the system is unstable for small signals (amplitude of oscillation increased). If the amplitude of oscillation increased beyond the point 5 then the system becomes stable. Similarly, if the system is actuated by a large disturbance then it produces a large amplitude signal so the point

* detail of limit cycle is given at the end of the chapter.
Example 7.4. For the non-linear system shown in the fig. 7.19 determine the amplitude and frequency of limit cycle. The describing function of non-linearity is given by

\[ G_D(x) = \frac{4M}{\pi X} \sin^{-1} \left( \frac{n}{X} \right) \]

Solution: Here \( M = 1 \) and \( n = 0.1 \)

\[ \therefore \quad G_D(x) = \frac{4}{\pi X} \sin^{-1} 0.1 \times \frac{1}{X} \]

\[ \frac{-1}{G_D(x)} = \frac{\pi X}{4} \left( 180^\circ + \sin^{-1} 0.1 \times \frac{1}{X} \right) \]

Plot the curve for eqn (7.16)

Table 7.1.

<table>
<thead>
<tr>
<th>X</th>
<th>-1/N ( \times 10^2 )</th>
<th>X</th>
<th>-1/N ( \times 10^2 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.1</td>
<td>0.0785 ( \leq 90^\circ )</td>
<td>0.5</td>
<td>0.393 ( \leq 168.5^\circ )</td>
</tr>
<tr>
<td>0.2</td>
<td>0.157 ( \leq 150^\circ )</td>
<td>0.8</td>
<td>0.628 ( \leq 172.8^\circ )</td>
</tr>
<tr>
<td>0.3</td>
<td>0.235 ( \leq 160.5^\circ )</td>
<td>1.0</td>
<td>0.785 ( \leq 174.3^\circ )</td>
</tr>
</tbody>
</table>

\[ G(s) = \frac{10}{(0.4s + 1)(2s + 1)} \]

put \( s = j\omega \)

\[ G(j\omega) = \frac{10}{(0.4\omega + 1)(1 + 2\omega)} \]

\[ |G(j\omega)| = \frac{10}{\sqrt{1 + 0.16\omega^2} \sqrt{1 + 4\omega^2}} \]

\[ \angle G(j\omega) = -\tan^{-1}0.4\omega - \tan^{-1}2\omega \]

Draw the polar plot for eqn 7.17

Table 7.2.

<table>
<thead>
<tr>
<th>( \omega )</th>
<th>( G(j\omega) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>3</td>
<td>1.052 ( \leq 130.7^\circ )</td>
</tr>
<tr>
<td>4</td>
<td>0.057 ( \leq 140.8^\circ )</td>
</tr>
<tr>
<td>6</td>
<td>0.319 ( \leq 152.6^\circ )</td>
</tr>
<tr>
<td>7</td>
<td>0.239 ( \leq 156.6^\circ )</td>
</tr>
<tr>
<td>8</td>
<td>0.186 ( \leq 159^\circ )</td>
</tr>
<tr>
<td>10</td>
<td>0.121 ( \leq 163.1^\circ )</td>
</tr>
<tr>
<td>20</td>
<td>0.031 ( \leq 171.4^\circ )</td>
</tr>
</tbody>
</table>

The curves of \( \frac{1}{G_D(X)} \) and \( G(j\omega) \) intersect at the point \( A \) which corresponds to stable limit cycle

\[ \frac{-1}{G_D(X)} = 0.21 \]

\[ \Rightarrow \frac{\pi X}{4} = 0.21 \]

\[ \Rightarrow X = \frac{0.21 \times 4}{\pi} = 0.267 \]

\[ \Rightarrow |G(j\omega)| = \frac{10}{\sqrt{1 + 0.16\omega^2} \sqrt{1 + 4\omega^2}} = 0.21 \]

\[ \Rightarrow \omega = 7.92 \text{ rad/sec.} \]

Thus, stable limit cycle corresponds to \( X = 0.267 \) and \( \omega = 7.92 \text{ rad/sec.} \)

Example 7.5. Determine the stability of the system shown in fig. 7.21.

Solution. The describing function is given by

\[ G_D(X) = \frac{2K}{\pi} \left[ \sin^{-1} \left( \frac{s}{X} \right) + \frac{s}{X} \sqrt{1 - \left( \frac{s}{X} \right)^2} \right] \]

\[ G(s) = \frac{10}{(1 + 0.4s)(1 + 2s)} \]

Here, \( s = 1, K = 1 \)

\[ G_D(X) = \frac{2}{\pi} \left[ \sin^{-1} \left( \frac{1}{X} \right) + \frac{1}{X} \sqrt{1 - \left( \frac{1}{X} \right)^2} \right] \]

\[ G(s) = \frac{10}{(1 + 0.4s)(1 + 2s)} \]
put \( s = j\omega \)

\[
G(j\omega) = \frac{10}{(1 + j/0.4\omega)(1 + j2\omega)}
\]

\[
|G(j\omega)| = \frac{10}{\sqrt{1 + 0.16\omega^2} \sqrt{1 + 4\omega^2}}
\]

\[
\dot{G}(j\omega) = -\tan^{-1} 0.4 \omega - \tan^{-1} 2\omega
\]

Draw the plot for \(-\frac{1}{G_D(X)}\) and \(G(j\omega)\) (shown in fig. 7.22). From the graph we find that the curve lies outside the \(G(j\omega)\) plot. Hence, the system is always stable.

![Graph showing stability](image)

**Example 7.6.** Determine the amplitude and frequency of the limit cycle of the non-linearity shown in fig. 7.23.

![Block diagram](image)

**Solution:** The describing function of an ideal relay is given by

\[
G_D(X, j\omega) = \frac{4M}{\pi X} \angle 0^\circ
\]

\[
-\frac{1}{G_D(X, j\omega)} = \frac{\pi X}{4} \angle -180^\circ
\]

\( M = 1 \) and angle = 0°

**Table 7.3.**

<table>
<thead>
<tr>
<th>( X )</th>
<th>(-\frac{1}{G_D(X)})</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.1</td>
<td>0.0785 \angle -180^\circ</td>
</tr>
<tr>
<td>0.2</td>
<td>0.157 \angle -180^\circ</td>
</tr>
<tr>
<td>0.3</td>
<td>0.257 \angle -180^\circ</td>
</tr>
<tr>
<td>0.5</td>
<td>0.393 \angle -180^\circ</td>
</tr>
<tr>
<td>0.8</td>
<td>0.628 \angle -180^\circ</td>
</tr>
<tr>
<td>1.0</td>
<td>0.785 \angle -180^\circ</td>
</tr>
<tr>
<td>2.0</td>
<td>1.57 \angle -180^\circ</td>
</tr>
<tr>
<td>2.5</td>
<td>1.97 \angle -180^\circ</td>
</tr>
</tbody>
</table>

\[
G(j\omega) = \frac{10}{j\omega(1 + j\omega)(2 + j\omega)} \frac{(G(j\omega))}{|G(j\omega)|} = \frac{10}{\omega \sqrt{1 + \omega^2} \sqrt{4 + \omega^2}}
\]

\[
G(j\omega) = -90^\circ - \tan^{-1} \omega - \tan^{-1} \omega/2
\]

**Table 7.4.**

<table>
<thead>
<tr>
<th>( \omega )</th>
<th>( G(j\omega) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.2</td>
<td>2.287 \angle -171.2^\circ</td>
</tr>
<tr>
<td>1.4</td>
<td>1.701 \angle -179.45^\circ</td>
</tr>
<tr>
<td>1.5</td>
<td>1.479 \angle -183.2^\circ</td>
</tr>
<tr>
<td>1.8</td>
<td>1.003 \angle -192.93^\circ</td>
</tr>
<tr>
<td>2.0</td>
<td>0.79 \angle -198.43^\circ</td>
</tr>
<tr>
<td>5.0</td>
<td>0.073 \angle -236.9^\circ</td>
</tr>
<tr>
<td>10</td>
<td>0.0098 \angle -253^\circ</td>
</tr>
</tbody>
</table>

Intersection of curves \( G(j\omega) \) and \(-\frac{1}{G_D(X, j\omega)}\) gives a point which corresponds to stable limit cycle.

Here,

\[
\frac{\pi X}{4} = 1.68
\]

\[
X = 2.14
\]

Also, intersection is at \(-180^\circ\)

\[
\Rightarrow -180^\circ = -90^\circ - \tan^{-1} \omega - \tan^{-1} 0.5\omega
\]

\( \omega = 1.41 \text{ rad/sec} \)

Thus, the stability limit cycle corresponds to

\[
X = 2.14 \text{ and } \omega = 1.41 \text{ rad/sec}
\]
7.5. THE PHASE PLANE TECHNIQUE

The phase plane method is a graphical method for solving the second order non-linear differential equations. This method also gives the information about the types of transient response for various initial conditions.

Consider the equation

\[ \frac{d^2x}{dt^2} + A \frac{dx}{dt} + Bx = 0 \]

If,

\[ \frac{dx}{dt} = \dot{x}, \quad \frac{d^2x}{dt^2} = \ddot{x} \]

then equation 7.19 becomes

\[ \ddot{x} + A \dot{x} + Bx = 0 \]

or,

\[ \frac{d\ddot{x}}{dx} \]

Divide both sides by \( \frac{dx}{dt} \) we get

\[ \frac{d\ddot{x}}{dx} = \frac{-(A \dot{x} + Bx)}{\dot{x}} \]

The coordinate plane corresponding to \( x \) and \( \dot{x} \) is called phase plane. The curve described in phase plane with respect to time is called phase trajectory. The phase trajectory can be plotted by graphical or analytical method. The family of trajectory for different initial conditions is known as phase-portrait. The phase portrait gives the information about the stability and limit cycle (if exists).

The equation 7.21 can be plotted directly as there are three variables \( \frac{d\ddot{x}}{dx} \), \( \dot{x} \) and \( x \). To get the relationship between \( \dot{x} \) and \( x \) there are several methods.

(i) For differential equation: get the solution and obtain \( x \) and then differentiate. This is to get \( \dot{x} \) and eliminate independent variables then draw a curve.

(ii) Put the equation in general form, integrate and plot the result.

(iii) Equation 7.21 is the general equation for phase plane, if a constant value \( f \) is selected for \( K \) the equation 7.21 becomes

\[ K = f(\dot{x}, x) \]
Integrate both the sides
\[ \int E \, dE = -\omega_n^2 \int E \, dE \]
\[ \dot{E}^2 = -\omega_n^2 E^2 + C^2 \]
\[ C^2 = \dot{E}^2 + \omega_n^2 E^2 \]

or,
\[ \left( \frac{E}{\omega_n} \right)^2 + E^2 = \left( \frac{c}{\omega_n} \right)^2 \]

Equation (7.25) is the equation of the circle with radius \( \frac{c}{\omega_n} \).

The trajectory is shown in the fig. 7.26. From Fig. 7.26 the system having sustained oscillations.

### 7.7. PHASE TRAJECTORY OF A SECOND ORDER SYSTEM USING METHOD OF ISOCLINES

Second order system is given by
\[ \dot{E} + 2\zeta \omega_n E + \omega_n^2 E = 0 \]

or,
\[ \frac{dE}{dt} + 2\zeta \omega_n \frac{dE}{dt} + \omega_n^2 E = 0 \]

Divide the equation 7.27 by \( \frac{dE}{dt} = \dot{E} \), we get
\[ \frac{d\dot{E}}{dt} + 2\zeta \omega_n \dot{E} + \omega_n^2 E = 0 \]

put \( \frac{d\dot{E}}{dt} = N = \tan \phi = \text{slope of the trajectory} \)
\[ N + 2\zeta \omega_n \dot{E} + \omega_n^2 E = 0 \]

or,
\[ \dot{E} \left( N + 2\zeta \omega_n \right) + \omega_n^2 E = 0 \]

Consider the following example.

Example 7.7. Draw the trajectory when \( \zeta = 0.5, \omega_n = 1 \text{ rad/sec. for unit step input.} \)

Solution: From equation 7.28

The trajectory is shown in the fig. 7.26. From Fig. 7.26 the system having sustained oscillations.

**Table 7.5**

<table>
<thead>
<tr>
<th>( \theta = \tan^{-1} \frac{1}{N+1} )</th>
<th>( \phi = \tan^{-1} N )</th>
<th>( N )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0(^\circ)</td>
<td>90(^\circ)</td>
<td>( \infty )</td>
</tr>
<tr>
<td>-15(^\circ)</td>
<td>69.89(^\circ)</td>
<td>2.73</td>
</tr>
<tr>
<td>:</td>
<td>:</td>
<td>:</td>
</tr>
<tr>
<td>:</td>
<td>:</td>
<td>:</td>
</tr>
</tbody>
</table>

**SECOND METHOD**

Draw a line parallel to first slope marker of first isocline. Let intersect at \( a_1 \). Find the mid point of \( a_1 b_1 \). Let this be \( b_1 \). From \( b_1 \), draw a line parallel to second slope marker. Let it cut at \( a_2 \), the point...
7.8. CALCULATION OF TIME FROM PHASE TRAJECTORY

For evaluation of time two methods are used

(a) We define

$$\dot{x} = \frac{dx}{dt}$$

or,

$$dt = \frac{dx}{\dot{x}}$$

Integrate both sides between two points $P_1$ and $P_2$

$$\int_{P_1}^{P_2} dt = \int_{x_1}^{x_2} \frac{1}{\dot{x}} dx$$

$$t_{P_1P_2} = \int_{x_1}^{x_2} \frac{1}{\dot{x}} dx$$  (7.30)

If a curve of $\frac{1}{\dot{x}}$, $x$ is plotted, the area between two limits $P_1$, $P_2$ will represent the time.

(b) Mark the points on phase trajectory. The points should be so chosen that the curve between two consecutive points could be approximated as a straight line, and read the average speed or velocity at the center between the two points and the corresponding change in $x$ i.e. $dx$ measured. Do the calculation for each segment (between two points) and calculate the total time upto any point.

Hence,

$$\Delta t = \frac{\Delta x}{\dot{x}_{\text{average}}}$$

and

$$T = \sum \Delta t$$

7.8. STABILITY FROM THE PHASE PLANE

The stability can be determined from the phase-plane trajectory near the point of equilibrium called singular points.

Consider the second order system

$$\frac{dx}{dt} = P(x,y)$$  (7.31)

$$\frac{dy}{dt} = Q(x,y)$$  (7.32)

$$\frac{dy}{dx} = \frac{Q(x,y)}{P(x,y)}$$  (7.33)

The singular points are those values of $x$ and $y$ satisfying

$P(x,y) = 0$

$Q(x,y) = 0$

In fig. 7.31 (a) the trajectory approaches the origin. It will appear as a spiral and this type of singularity is known as Focus. Hence it is a stable focus. If the trajectory diverge from the focus then this will be unstable focus (fig. 7.31b).
7.10. AUTONOMOUS SYSTEM

Consider a non-linear system represented by the equation 7.34

\[ \dot{x} = f(x, u) \]  

where \( u(t) \) is the input vector and \( f \) is the non-linear function of \( x \) and \( u \).

If \( u(t) \) is identically zero, the system is known as an autonomous system and represented by equation (7.35)

\[ \dot{x} = f(x) \]  

The autonomous system is also known as an unforced system.

The points at which the derivatives of all state variables are zero, called the points of equilibrium (also called singular points), i.e. \( \dot{x} = 0 \)

7.11. ASYMPTOTIC STABILITY

Consider a hypersphere of finite radius surrounding the point of equilibrium is described by the equation (7.37)

\[ x_1^2 + x_2^2 + \ldots + x_n^2 = R^2 \]  

The system is said to be asymptotically stable if there exists a \( R > 0 \) such that any trajectory starting from any point within \( s(R) \) does not leave \( s(R) \) at any time and returns to the origin (shown in fig. 7.33).

Fig. 7.34.

The system represented by the equation 7.35 is asymptotically stable in the large if it is asymptotically stable for all states from which trajectories originate.

7.12. LIMIT CYCLES

The definition of stability in the sense of Lyapunov includes the possibility of the state of a perturbed non-linear system following a closed trajectory within the tolerance limits specified by the region of stability. This behaviour is called limit cycle. The limit cycle describes the oscillations of non-linear systems. The limit cycle corresponds to an oscillation of fixed amplitude and period but not necessarily sinusoidal.

Consider van der Pol's differential equation

\[ \frac{d^2 x}{dt^2} + \mu (1 - x^2) \frac{dx}{dt} + x = 0 \]  

Compare the equation 7.38 with linear differential equation

\[ \frac{d^2 x}{dt^2} + 2 \zeta \frac{dx}{dt} + \omega_0^2 x = 0 \]  

If \( \mu >> 1 \), then damping factor has positive value. Therefore the system behaves like overdamped system with decrease of amplitude of \( x \), (shown by the outer trajectory fig. 7.34).

If \( \mu << 1 \), then the damping factor has negative value with increase of amplitude of \( x \) till the system state again enters the limit cycle (shown by the inner trajectory fig. 7.34).

A non-linear system does not possess the homogeneity and superposition properties.

If the non-linearities are inherently present in the system then these are known as incidental non-linearities. If some non-linearities are intentionally added to the system, then such types of non-linearities are called intentional non-linearities.
Some common type of non-linearities are saturation, dead zone, friction etc.

The describing function is defined as the ratio of amplitudes and phase angle between the fundamental components of the output and input sinusoid, for all amplitudes of the input which are to be considered and for all frequencies from zero to +∞.

\[ G_D(j\omega) = \frac{\text{Magnitude and phase angle of fundamental component of output}}{\text{magnitude and phase angle of input}} \]

Describing function for saturation is given by

\[ G_D(j\omega) = \frac{2K}{\pi} \sin^{-1}R + R\sqrt{1 - R^2} e^{j0^\circ} \]

where

\[ R = \frac{E_o}{E} \]

Describing function for ideal relay is

\[ G_D(j\omega) = \frac{4Y}{\pi x} e^{j0^\circ} \]

Describing function for practical relay

\[ G_D(j\omega) = \frac{4Y}{\pi x} \left[ 1 - \frac{\omega^2}{4x^2} \right]^{1/2} e^{j0^\circ} \]

Describing function for the combination of dead zone and saturation

\[ G_D(j\omega) = \begin{cases} 0 & ; x < \frac{D}{2}, \alpha = \beta = \pi/2 \\ \left[ 1 - \frac{2}{\pi} \left( \alpha + \sin \alpha \cos \alpha \right) \right] k & ; \frac{D}{2} < x < s \\ \frac{k}{\pi} \left[ 2(\beta - \alpha) + \sin 2\beta - \sin 2\alpha \right] & ; x > s \end{cases} \]

Describing function of backlash is given by

\[ G_D(j\omega) = \frac{k}{\pi} \left[ \frac{\pi}{2} + \sin^{-1}(1-2R) \right]^2 + 4R(1-R) + \left[ \pi + 2\sin^{-1}(1-2R) \right] 2R(1-2R) \]

\[ \frac{G_D(j\omega)}{\omega} = \tan^{-1} \frac{4R(R-1)}{\pi/2 + \sin^{-1}(1-2R) + 2R(1-2R)} \left| \frac{1-R}{R} \right| \]

The phase plane method is a graphical method for solving the second order non-linear differential equations. This method gives the information about the types of transient response for various initial conditions.
State Space Analysis of Control System

8.1. ANALYSIS OF SYSTEMS
The procedure for determining the state of a system is called state variable analysis. The state of a dynamic system is the smallest set of variables such that the knowledge of the input for any time t ≥ t₀ completely determines the behaviour of the system for all time t ≥ t₀. This set of variables is called state variables.

In earlier chapters we studied the linear system by transfer function, block diagram etc. The transfer function has some drawbacks e.g. transfer function is only defined under zero initial conditions and also it is applicable to linear time invariant systems. Therefore due to these limitations the state variable approach is developed. This technique can be used for analysis and design of linear, nonlinear, time invariant or time variant and multi input multi output systems. The state space analysis involves the description of the system in terms of first order differential equations by selecting suitable state variables, the first order derivatives are arranged on left hand side and on right hand side the terms are free from derivatives. The state space techniques have many advantages (See in next article i.e. 8.2).

8.2. ADVANTAGES OF STATE SPACE TECHNIQUES
This technique has the following advantages.
1. This approach can be applied to linear or nonlinear, time variant or time invariant systems.
2. It is easier to apply where the Laplace transform cannot be applied.
3. Order differential equations can be expressed as equation of first order whose solutions are easier.
4. It is a time domain approach.
5. This method is suitable for digital computer computation because this is a time domain approach.
6. The system can be designed for optimal conditions with respect to given performance index.

8.3. SOME IMPORTANT DEFINITIONS
State: The state of a system at any time t ≥ t₀ is the minimum set of numbers x₁, x₂, ..., xₙ which along with the input to the system for time t ≥ t₀ is sufficient to determine the behaviour of the system for all t ≥ t₀. In other words, the state of a system represents the minimum amount of information that we need to know about a system at t₀ such that its future behaviour can be determined with reference to the input before t₀. The state can also be defined as the state of a system at time t₀.

State space: The n-dimensional space whose coordinate axes consists of the x₁ axis, x₂ axis, ..., xₙ axis is called state space. Any state can be represented by a point in the state space.

8.4. STATE SPACE REPRESENTATION
8.4.1. State Representation for Electrical Network (Physical Variable Form)
Consider an RLC network shown in fig. 8.2. Let, the current at time t = 0 be i₁(0) and capacitor voltage at time t = 0 be Vc(t₀) = 0. Thus, the state of the network at time t = 0 is specified by the inductor current and capacitor voltage. Hence, the pair i₁(0), Vc(t₀) is called the initial state of the network.

Similarly at time t = t', the pair i₁(t'), Vc(t') is called the state of the network at t'. The variable i₁ and Vc are called state variables of the network.

Apply KVL
\[ R_i + L \frac{di_1}{dt} + V_c = 0 \]  \( \text{(8.1)} \)

Also,
\[ i_c = i_L = C \frac{dV_c}{dt} \]  \( \text{(8.2)} \)

from (8.1)
\[ \frac{di_1}{dt} = -\frac{R}{L} i_1 - \frac{1}{L} V_c \]  \( \text{(8.3)} \)
\[ \frac{dv_c}{dt} = \frac{1}{C} i_1 \]  \( \text{(8.4)} \)

Equations of this form are called state equations. In such equations all the variables present are state variables.

Equations 8.3 and 8.4 can be written in matrix form as
\[
\begin{bmatrix}
\frac{di_1}{dt} \\
\frac{dv_c}{dt}
\end{bmatrix} =
\begin{bmatrix}
\frac{-R}{L} & -\frac{1}{L} \\
\frac{1}{C} & 0
\end{bmatrix}
\begin{bmatrix}
i_1 \\
v_c
\end{bmatrix}
\]  \( \text{(8.5)} \)
let \( x(t) = \begin{bmatrix} i(t) \\ V_C \end{bmatrix} \) and \( A = \begin{bmatrix} R & -1 \\ L & L \\ \frac{1}{C} & 0 \end{bmatrix} \)

then equation 8.5 can be written as
\[
\frac{d}{dt} x(t) = A x(t)
\]
or,
\[
\dot{x}(t) = A x(t)
\]
In the linear time-invariant systems, the general form of state equations are
\[
\begin{align*}
\dot{x}(t) &= A x(t) + B u(t) \\
y(t) &= C x(t) + D u(t)
\end{align*}
\]
These equations are vector differential equations where \( \dot{x} \) is the \( n \)-dimensional state vector, \( y = n \)-dimensional output vector, \( u = r \)-dimensional control vector or input vector, \( A = n \times n \) system matrix, \( B = n \times r \) control matrix, \( C = n \times n \) output matrix.

In some cases there is no direct connection between input and output so \( D u(t) \) will not be there
\[
y(t) = C x(t)
\]

Equation (8.6) and (8.8) can be expressed as
\[
\begin{bmatrix}
x_1 \\
x_2 \\
x_3 \\
\vdots \\
x_n
\end{bmatrix} = \begin{bmatrix}
a_{11} & a_{12} & \cdots & a_{1n} \\
a_{21} & a_{22} & \cdots & a_{2n} \\
a_{31} & a_{32} & \cdots & a_{3n} \\
\vdots & \vdots & \ddots & \vdots \\
a_{n1} & a_{n2} & \cdots & a_{nn}
\end{bmatrix} \begin{bmatrix}
x_1 \\
x_2 \\
x_3 \\
\vdots \\
x_n
\end{bmatrix} + \begin{bmatrix}
b_{11} & b_{12} & \cdots & b_{1r} \\
b_{21} & b_{22} & \cdots & b_{2r} \\
b_{31} & b_{32} & \cdots & b_{3r} \\
\vdots & \vdots & \ddots & \vdots \\
b_{n1} & b_{n2} & \cdots & b_{nr}
\end{bmatrix} \begin{bmatrix}
u_1 \\
u_2 \\
u_3 \\
\vdots \\
u_r
\end{bmatrix}
\]
\[
\begin{bmatrix}
y_1 \\
y_2 \\
y_3 \\
\vdots \\
y_n
\end{bmatrix} = \begin{bmatrix}
C_{11} & C_{12} & \cdots & C_{1m} \\
C_{21} & C_{22} & \cdots & C_{2m} \\
C_{31} & C_{32} & \cdots & C_{3m} \\
\vdots & \vdots & \ddots & \vdots \\
C_{n1} & C_{n2} & \cdots & C_{nm}
\end{bmatrix} \begin{bmatrix}
x_1 \\
x_2 \\
x_3 \\
\vdots \\
x_n
\end{bmatrix}
\]

\[\text{Example 8.2.} \quad \text{A system is described by the following differential equation. Represent the system in state space.}
\]
\[
d\frac{dx}{dt} + 3 \frac{d^2x}{dt^2} + 4 \frac{dx}{dt} + 4x = u_1(t) + 3u_2(t) + 4u_3(t)
\]

\[\text{Output} \quad y(t) = \frac{dx}{dt} + 3u_1 \]

4.2(b): State space representation of \( n \)-th order Linear system with \( r \) Forcing function

Consider the following example

Example 8.2. A system is described by the following differential equation. Represent the system in state space.
\[
\frac{d^2x}{dt^2} + 3 \frac{d^2x}{dt^2} + 4 \frac{dx}{dt} + 4x = u_1(t) + 3u_2(t) + 4u_3(t)
\]

\[\text{Output} \quad y(t) = \frac{dx}{dt} + 3u_1 \]

8.4.2. State Space Representation of \( n \)-th Order Differential Equations

Consider the following examples

8.4.2(a) For \( n \)-th order differential equation

Example 8.1. A system is described by the differential equation
\[
d^3y \over dt^3 + 6 \frac{d^2y}{dt^2} + 11 \frac{dy}{dt} + 10y = 8u(t)
\]

where \( y \) is the output and \( u \) is the input to the system. Obtain state space representation of the system.

Solution: Select the state variables as

\[
\begin{bmatrix}
x_1 \\
x_2 \\
x_3 \\
\vdots \\
x_n
\end{bmatrix} = \begin{bmatrix}
x(t) \\
x(t) \\
x(t) \\
\vdots \\
x(t)
\end{bmatrix}
\]

\[
\begin{bmatrix}
x_1 \\
x_2 \\
x_3 \\
\vdots \\
x_n
\end{bmatrix} = \begin{bmatrix}
x_1 \\
x_2 \\
x_3 \\
\vdots \\
x_n
\end{bmatrix}
\]

\[
\begin{bmatrix}
y_1 \\
y_2 \\
y_3 \\
\vdots \\
y_n
\end{bmatrix} = \begin{bmatrix}
y(t) \\
y(t) \\
y(t) \\
\vdots \\
y(t)
\end{bmatrix}
\]

\[
\begin{bmatrix}
x_1 \\
x_2 \\
x_3 \\
\vdots \\
x_n
\end{bmatrix} = \begin{bmatrix}
x_1 \\
x_2 \\
x_3 \\
\vdots \\
x_n
\end{bmatrix}
\]

\[
\begin{bmatrix}
y_1 \\
y_2 \\
y_3 \\
\vdots \\
y_n
\end{bmatrix} = \begin{bmatrix}
y(t) \\
y(t) \\
y(t) \\
\vdots \\
y(t)
\end{bmatrix}
\]
8.4.3. State Space Representation for Transfer Function

Consider the following example

Example 8.3. For the given transfer function, obtain the state model.

\[ G(s) = \frac{y(s)}{u(s)} = \frac{K}{s^3 + a_3 s^2 + a_2 s + a_1} \]

Solution: This transfer function has no zeros.

\[ (s^3 + a_3 s^2 + a_2 s + a_1) y(s) = Ku(s) \]

or,

\[ s^3 y(s) + a_3 s^2 y(s) + a_2 s y(s) + a_1 y(s) = Ku(s) \]

Taking inverse laplace

\[ \frac{\dot{y}(t)}{u(t)} = Ku(t) \]

or

\[ \frac{\ddot{y}(t)}{u(t)} = Ku(t) \]

Select the state variables as, first state variable as output

\[ \begin{align*}
    y(t) &= x_1 \\
    \dot{y}(t) &= x_2 \\
    \ddot{y}(t) &= x_3 \\
    \vdots \\
    \dot{y}(t) &= x_3 \\
    y(t) &= \begin{bmatrix} 0 & 1 & 0 & \vdots & 0 \\
    0 & 1 & \ddots & \vdots & \vdots \\
    -a_3 & -a_2 & \ddots & 0 \\
    -a_1 & -a_2 & \cdots & 0 \\
    \end{bmatrix} \begin{bmatrix} x_1 \\
    x_2 \\
    x_3 \\
    \vdots \\
    x_3 \\
    \end{bmatrix} + \begin{bmatrix} 0 \\
    0 \\
    0 \\
    0 \\
    K \end{bmatrix} u(t)
\end{align*} \]

Rewriting the equations

\[ \begin{align*}
    \dot{x}_1 &= x_2 \\
    \dot{x}_2 &= x_3 \\
    \dot{x}_3 &= -a_3 x_3 - a_2 x_2 - a_1 x_1 + Ku(t)
\end{align*} \]

BLOCK DIAGRAM:

The block diagram of the given transfer function is shown in fig. 8.3.

Now consider another case when the transfer function has zeros.

Example 8.4. Obtain the state model for the given transfer function.

\[ G(s) = \frac{y(s)}{u(s)} = \frac{K(C_2 s + C_1)}{s^3 + a_3 s^2 + a_2 s + a_1} \]
8.5. SOLUTION OF THE TIME-INVARIANT STATE EQUATION

8.5.1. Solution of Homogeneous State Equation: Laplace Transform Method

We know that

\[ \dot{x}(t) = Ax(t) + Bu(t) \]

\[ u(t) = 0 \quad \text{for unforced response} \]

then

\[ \dot{x}(t) = Ax(t) \]

Let us consider the analogous scalar equation

\[ \dot{x}(t) = ax(t) \]

Take the Laplace transform of equation 8.14

\[ sX(s) - x(0) = ax(s) \]

\[ (s - a) X(s) = x(0) \]

or,

\[ X(s) = (s - a)^{-1} x(0) \]

Take the inverse Laplace of equation 8.15

\[ x(t) = e^{at} x(0) \]

If equation (8.16) is the solution of equation (8.14) then the solution of equation (8.13)

\[ x(t) = e^{At} x(0) \]

\[ e^{At} = \phi(t) = \text{State Transition Matrix (STM)} \]

\[ = I + At + \frac{A^2 t^2}{2!} + \frac{A^3 t^3}{3!} + \ldots = \sum_{i=0}^{\infty} \frac{A^i t^i}{i!} \]

\[ \phi(t) = e^{At} = \mathcal{L}^{-1} \phi(s) = \mathcal{L}^{-1} [sI - A]^{-1} \]

where \( \phi(s) = \text{Resolvant matrix} \)

8.5.2. Properties of State Transition Matrices

For time-invariant system:

\[ \dot{x} = Ax \]

\[ \phi(t) = e^{At} = I + At + \frac{A^2 t^2}{2!} + \frac{A^3 t^3}{3!} + \ldots \]

(i) \( \phi(0) = e^{A0} = I \)

(ii) \( \phi(t) = e^{At} = (e^{-At})^{-1} = [\phi(-t)]^{-1} \)

(iii) \( \phi^{-1}(t) = \phi(-t) \)

(iv) \( \phi(t_1 + t_2) = e^{A(t_1 + t_2)} = e^{At_1} e^{At_2} = \phi(t_1) \phi(t_2) \phi(t_2) \phi(t_1) \)

(v) \( [\phi(t)]^{1/n} = \phi(nt) \)

(vi) \( \phi(t_2 - t_1) \phi(t_1 - t_0) = \phi(t_2 - t_0) = \phi(t_1 - t_0) \phi(t_2 - t_1) \)

8.5.3. Solution of Non-Homogeneous State Equations

Here \( u(t) \neq 0 \)

a. Initial condition \( t = 0 \)

\[ \dot{x}(t) = Ax(t) + Bu(t) \]

Consider an analogous scalar equation

\[ \dot{x}(t) = ax(t) + bu(t) \]

where 'a' and 'b' are the scalar quantities. Take Laplace transform of equation (8.19).

\[ sX(s) - x(0) = aX(s) + bu(s) \]

Multiply both the sides by \( (s - a)^{-1} \) we get

\[ X(s) = (s - a)^{-1} x(0) + (s - a)^{-1} bu(s) \]

Take inverse Laplace of equation (8.20)

\[ x(t) = e^{At} x(0) + \int_{0}^{t} e^{A(t-\tau)} Bu(\tau) d\tau \]

Equation (8.21) is the solution of equation (8.19), then the solution of equation (8.18) will be

\[ x(t) = e^{At} x(0) + \int_{0}^{t} e^{A(t-\tau)} Bu(\tau) d\tau \]

\[ \therefore \]

\[ x(t) = \phi(t) x(0) + \int_{0}^{t} \phi(t - \tau) Bu(\tau) d\tau \]

If initial time is given by \( t_0 \), i.e. \( t = t_0 \)

\[ x(t_0) = \phi(t_0) x(0) + \int_{0}^{t_0} \phi(t_0 - \tau) Bu(\tau) d\tau \]

\[ \therefore \]

\[ x(t_0) = \phi(t_0) x(0) + \int_{t_0}^{0} \phi(t - \tau) Bu(\tau) d\tau \]

\[ \therefore \]

\[ x(t_0) = \phi(t_0) x(0) + \int_{0}^{t_0} \phi(t_0 - \tau) Bu(\tau) d\tau \]

\[ \therefore \]

\[ x(t_0) = \phi(t_0) x(0) + \int_{0}^{t_0} \phi(t - \tau) Bu(\tau) d\tau \]

\[ \therefore \]

\[ x(t_0) = \phi(t_0) x(0) + \int_{0}^{t_0} \phi(t - \tau) Bu(\tau) d\tau \]

\[ \therefore \]

\[ x(t_0) = \phi(t_0) x(0) + \int_{0}^{t_0} \phi(t - \tau) Bu(\tau) d\tau \]

\[ \therefore \]

\[ x(t_0) = \phi(t_0) x(0) + \int_{0}^{t_0} \phi(t - \tau) Bu(\tau) d\tau \]

\[ \therefore \]

\[ x(t_0) = \phi(t_0) x(0) + \int_{0}^{t_0} \phi(t - \tau) Bu(\tau) d\tau \]

\[ \therefore \]

\[ x(t_0) = \phi(t_0) x(0) + \int_{0}^{t_0} \phi(t - \tau) Bu(\tau) d\tau \]

\[ \therefore \]

\[ x(t_0) = \phi(t_0) x(0) + \int_{0}^{t_0} \phi(t - \tau) Bu(\tau) d\tau \]
Equation (8.26) can also be written as
\[ x(t) = e^{At}x(0) + \int_0^t e^{A(t-\tau)}Bu(\tau)\,d\tau \]

Example 8.5. Find the time response of the system described by the equation
\[ \dot{x}(t) = \begin{bmatrix} -1 & 1 \\ 0 & -2 \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t) \]
\[ x(0) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, u(t) = 1, \ t > 0 \]

Solution:
\[ x(t) = \phi(t) x(0) + \int_0^t \phi(t-\tau) Bu(\tau)\,d\tau \]

Step 1: Calculation of \( \phi(t) \):
\[ \phi(t) = L^{-1}\phi(s) = L^{-1}(sI - A)^{-1} \]
\[ \phi(s) = [sI - A]^{-1} \]
\[ sI - A = \begin{bmatrix} s & 0 \\ 0 & s \end{bmatrix} - \begin{bmatrix} -1 & 1 \\ 0 & -2 \end{bmatrix} = \begin{bmatrix} s+1 & -1 \\ 0 & s+2 \end{bmatrix} \]
\[ [sI - A]^{-1} = \begin{bmatrix} \frac{s+1}{s^2 + 3s + 2} & -\frac{1}{s^2 + 3s + 2} \\ 0 & \frac{s+1}{s^2 + 3s + 2} \end{bmatrix} \]
\[ \phi(t) = L^{-1}\phi(s) = \begin{bmatrix} \frac{e^t}{s^2 + 3s + 2} & -\frac{e^t}{s^2 + 3s + 2} \\ 0 & \frac{e^t}{s^2 + 3s + 2} \end{bmatrix} \]

Put all the values in equation (8.28)
\[ x(t) = \begin{bmatrix} e^t & e^t - e^{2t} \\ 0 & e^{2t} \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} + \int_0^t \begin{bmatrix} e^{-(t-\tau)} & e^{-(t-\tau)} - e^{-2(t-\tau)} \\ 0 & e^{-2(t-\tau)} \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} \,d\tau \]
\[ = \begin{bmatrix} -e^t \, e^{-2t} \\ 0 \end{bmatrix} + \int_0^t \begin{bmatrix} e^{-(t-\tau)} & e^{-(t-\tau)} - e^{-2(t-\tau)} \\ 0 & e^{-2(t-\tau)} \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} \,d\tau \]
\[ x_1(t) = -e^t + \int_0^t e^{-(t-\tau)} - e^{-2(t-\tau)} \,d\tau \]
\[ x_2(t) = \int_0^t e^{-2(t-\tau)} \,d\tau \]

Example 8.6. The system equations are given by
\[ \dot{x}(t) = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t) \]
\[ y(t) = \begin{bmatrix} 1 & 0 \end{bmatrix} x(t) \]

Find the transfer function of the system.
Solution: Given that
\[ A = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix}, B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, C = \begin{bmatrix} 1 & 0 \end{bmatrix} \]
8.7. COMPUTATION OF STATE TRANSITION MATRIX : $e^{At}$:

8.7.1. Laplace Transform Method

We know that

$$\dot{x}(t) = Ax + Bu(t)$$

For unforced system $u(t) = 0$

$$\dot{x}(t) = Ax$$

Take the Laplace transform

$sX(s) - x(0) = AX(s)$

or

$$[sI - A]X(s) = x(0)$$

Multiply both the sides by $[sI - A]^{-1}$

$$X(s) = [sI - A]^{-1} x(0)$$

$$[sI - A]^{-1} = \phi(s) = \text{Resolvent matrix}$$

$$\phi(t) = e^{At} = L^{-1}[sI - A]^{-1}$$

Where, $\phi(t) = e^{At} = \text{state transition matrix}$

Example 8.7. Compute the STM when

$$A = \begin{bmatrix} -1 & 1 \\ 0 & 2 \end{bmatrix}$$

Solution:

$$[sI - A] = \begin{bmatrix} s & 0 \\ 0 & s \end{bmatrix} - \begin{bmatrix} -1 & 1 \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} s+1 & -1 \\ 0 & s-2 \end{bmatrix}$$

$$[sI - A]^{-1} = \begin{bmatrix} \frac{1}{s^2 + 3s + 2} & \frac{s+1}{s^2 + 3s + 2} \\ \frac{s-2}{s^2 + 3s + 2} & \frac{1}{s^2 + 3s + 2} \end{bmatrix}$$

Hence, required transfer function = \( \frac{1}{s^2 + 3s + 2} \)  Ans.

8.7.2. Series Summation Method

$$\phi(t) = e^{At} = I + At + \frac{A^2t^2}{2!} + ...$$

Example 8.8. Evaluate the STM by series summation method

$$A = \begin{bmatrix} 2 & -2 & 3 \\ 1 & 1 & 1 \\ 1 & 3 & -1 \end{bmatrix}$$

Solution: Given that

$$A^2 = \begin{bmatrix} 5 & 3 & 1 \\ 14 & 4 & 17 \\ 13 & 3 & 11 \end{bmatrix}, A^3 = \begin{bmatrix} 4 & 2 & 7 \\ 13 & 11 & 3 \end{bmatrix}$$

and so on.

$$e^{At} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} + t \begin{bmatrix} 2t & -2t & 3t \\ t & t & 2t \\ 2 & 4 & 2 \end{bmatrix} \frac{t^2}{2} + \begin{bmatrix} 14 & -4 & 17 \\ 13 & 3 & 11 \end{bmatrix} \frac{t^3}{6} + ...$$

8.7.3 Eigen Values

The roots of the characteristic equation are known as eigen values of matrix.

8.7.4 Eigen Vectors

Any non zero vector $P$, that satisfies the matrix equation $(\lambda I - A)P = 0$ where, $\lambda$ is $i^{th}$ eigen value of $A$ is known as eigen vector of $A$ associated with the eigen value $\lambda$.

8.7.5 Use of Diagonal

Let

$$A = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix}$$
\[ e^{At} = I + At + \frac{A^2t^2}{2!} + \ldots \]

\[
\begin{bmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1 \\
\end{bmatrix}
+ \begin{bmatrix}
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0 \\
\end{bmatrix}
= \begin{bmatrix}
1 & \lambda_1 & \lambda_2 t \\
0 & 1 & \lambda_2 \\
0 & 0 & 1 \\
\end{bmatrix}
\]

let \( \lambda \) is the eigen value of \( A \) and \( p_i \) is the eigenvector, then for each of the \( n \) eigen values of \( A \)

\[
AP_i = \lambda_i P_i \\
AP_1 = \lambda_1 P_1 \\
\vdots \\
AP_n = \lambda_n P_n \\
AM = M^\wedge \\
\]

Where \( \wedge \) = diagonal matrix with eigen values on its main diagonal

\[
M = \text{modal matrix} \\
\wedge = M^{-1}AM \\
e^A = M e^{\wedge} M^{-1} \\
\]

Example 8.9. Compute \( e^{At} \) when

\[
A = \begin{bmatrix}
0 & 1 \\
-3 & -4 \\
\end{bmatrix}
\]

Solution :

\[
\lambda I - A = \begin{bmatrix}
1 & 0 \\
0 & 1 \\
\end{bmatrix} = \begin{bmatrix}
\lambda & -1 \\
0 & \lambda - 4 \\
\end{bmatrix}
\]

\[
|\lambda I - A| = \lambda^2 + 4\lambda + 3 = 0 \\
\lambda_1 = -1 \\
\lambda_2 = -3
\]

let the eigenvector

\[
P_1 = \begin{bmatrix}
p_{11} \\
p_{21} \\
\end{bmatrix} \text{ and } P_2 = \begin{bmatrix}
p_{12} \\
p_{22} \\
\end{bmatrix}
\]

Put \( \lambda = -1 \) in equation (8.43)

\[
\begin{bmatrix}
-1 & -1 \\
3 & 3 \\
\end{bmatrix}
\begin{bmatrix}
p_{11} \\
p_{21} \\
\end{bmatrix} = \begin{bmatrix}
0 \\
0 \\
\end{bmatrix}
\]

\[
- p_{11} - p_{21} = 0 \\
3p_{11} + 3p_{21} = 0
\]

let

\[
p_{11} = 1 \text{ then } p_{21} = -1 \\
\]

Put

\[
\begin{bmatrix}
-3 & -1 \\
3 & 1 \\
\end{bmatrix}
\begin{bmatrix}
p_{12} \\
p_{22} \\
\end{bmatrix} = \begin{bmatrix}
0 \\
0 \\
\end{bmatrix}
\]

1.8 BLOCK DIAGRAM OF A LINEAR SYSTEM IN STATE VARIABLE FORM

1.8.1 MIMO System

The system is represented by the following equations

\[
\dot{x} = Ax + Bu \\
y = Cx + Du
\]

The block diagram is shown in the fig. 8.5.

1.8.2 Single Input Single Output System

\[
\dot{x}(t) = Ax + bu \\
y(t) = C^T x + du
\]

The block diagram is shown in fig 8.6.
8.9. CONTROLLABILITY AND OBSERVABILITY

A system is said to be controllable if any initial state \( x(t_0) \) or \( x_0 \) can be transferred to any final state \( x(t) \) in a finite time interval \( (t_1 - t_0) \), \( t \geq 0 \) by some control \( u \).

A system is said to be observable if every state \( x_0 \) can be exactly determined from the measurement of the output \( y \) over a finite interval of time \( 0 \leq t \leq t_p \).

The concept of controllability and observability were introduced by Kalman and they play an important role in the design of control systems in state space.

There are four possible states for the system.

(i) The system is controllable and observable
(ii) The system is controllable and unobservable
(iii) The system is uncontrollable and observable
(iv) The system is uncontrollable and unobservable

Consider the following example to determine the controllability and observability.

8.9.1. With the Help of Block Diagram

Example 8.10. Determine the controllability and observability of the system described by the state equation. Find the transfer function and draw the block diagram.

\[
\begin{align*}
    \dot{x}(t) &= \begin{bmatrix}
        -1 & 0 & 0 \\
        0 & -2 & 0 \\
        0 & 0 & -3
    \end{bmatrix} x(t) + \begin{bmatrix}
        1 \\
        1 \\
        0
    \end{bmatrix} u(t) \\
    y(t) &= \begin{bmatrix}
        1 & 0 & +2
    \end{bmatrix} x(t)
\end{align*}
\]

Solution:

\[
\begin{align*}
    \dot{x}(t) &= \begin{bmatrix}
        -1 & 0 & 0 \\
        0 & -2 & 0 \\
        0 & 0 & -3
    \end{bmatrix} x(t) + \begin{bmatrix}
        1 \\
        1 \\
        0
    \end{bmatrix} u(t) \\
    \dot{x}_1 &= -x_1 + u \\
    \dot{x}_2 &= -2x_2 + u \\
    \dot{x}_3 &= -3x_3 \\
    y &= x_1 + 2x_3
\end{align*}
\]

The block diagram is shown in Fig. 8.7.

From the block diagram, the first state is controllable and observable, second state is controllable and unobservable (scu), the third state is uncontrollable and observable (uso). Since all the states are not controllable and observable the whole system is uncontrollable and unobservable.

The transfer function is available only for the state which is both controllable and observable. Hence, the required transfer function is

\[
    G(s) = \frac{1}{s^2 + 4s + 1}
\]

Test for controllability and observability.

Give that

\[
    A = \begin{bmatrix}
        -0.5 & 0 \\
        0 & -2
    \end{bmatrix}, \quad B = \begin{bmatrix}
        0 \\
        1
    \end{bmatrix}
\]

\[
    AB = \begin{bmatrix}
        -0.5 & 0 \\
        0 & -2
    \end{bmatrix} \begin{bmatrix}
        0 \\
        1
    \end{bmatrix} = \begin{bmatrix}
        0 \\
        0
    \end{bmatrix}
\]

\[
    Q = \begin{bmatrix}
        0 & 0 \\
        0 & 1
    \end{bmatrix}
\]

Rank of \( Q \) is one. Hence system is uncontrollable.*

\[
    C = \begin{bmatrix}
        0 & 1
    \end{bmatrix}, \quad C^T = \begin{bmatrix}
        0 \\
        1
    \end{bmatrix}
\]

\[
    A^T = \begin{bmatrix}
        -0.5 & 0 \\
        0 & -2
    \end{bmatrix}, \quad A^T C^T = \begin{bmatrix}
        -0.5 & 0 \\
        0 & -2
    \end{bmatrix} \begin{bmatrix}
        0 \\
        1
    \end{bmatrix} = \begin{bmatrix}
        0 \\
        -2
    \end{bmatrix}
\]

\[
    Q' = [C^T : A^T C^T]
\]

\[
    Q' = \begin{bmatrix}
        0 & 0 \\
        0 & 1
    \end{bmatrix}
\]

Rank of \( Q' \) is one. Hence the system is unobservable. For controllability and observability the rank should be two.

The given system is uncontrollable and unobservable.
8.10. TIME VARYING SYSTEM

Consider the \( n \)-dimensional linear time-varying dynamical equation
\[
\begin{align*}
\dot{x}(t) &= A(t) x(t) + B(t) u(t) \\
y(t) &= C(t) x(t) + D(t) u(t)
\end{align*}
\]

Where,
\( x(t) \) is an \( n \) column vector, \( A(t) \) and \( B(t) \) are respectively \( n \times n \) matrix and \( n \times m \) matrix whose elements are function of time, \( u(t) \) is \( m \) column vector of forcing function.

The solution of equation (8.45) with \( x(t_0) = x_0 \) is given by
\[
x(t) = \phi(t, t_0) x_0 + \int_{t_0}^{t} \phi(t, \tau) B(\tau) u(\tau) d\tau
\]

Where \( \phi(t, \tau) \) is the state transition matrix of \( \dot{x} = A(t) x \), or, equivalently, the unique solution of
\[
\frac{d}{dt} \phi(t, \tau) = A(t) \phi(t, \tau) \quad \phi(t, t) = I
\]

Equation (8.47) is obtained from the equation (8.46) by using
\[
\phi(t, t_0) = \phi(t, t) \phi(t, t_0)
\]
\[
\phi(t, t_0) = I + \int_{t_0}^{t} A(t_1, t_0) dt_1 + \int_{t_0}^{t} A(t_1) \int_{t_0}^{t_1} A(t_2) dt_2 dt_1 + ...
\]

Example 8.12. Determine the state transition matrix \( \phi(t, 0) \) for the system
\[
\dot{x} = \begin{bmatrix} 0 & 1 \\ t & 0 \end{bmatrix} x
\]

Solution : \( t_0 = 0 \)
\[
\int_{0}^{t} A(t_1) dt = \int_{0}^{t} \begin{bmatrix} 0 & 1 \\ t & 0 \end{bmatrix} dt = \begin{bmatrix} 0 & t \\ t^2 / 2 & 0 \end{bmatrix}
\]
\[
\int_{0}^{t} A(t_1) \int_{0}^{t_1} A(t_2) dt_2 dt_1 = \int_{0}^{t} \begin{bmatrix} 0 & 1 \\ t & 0 \end{bmatrix} \int_{0}^{t_1} \begin{bmatrix} 0 & 0 \\ t_1 & 0 \end{bmatrix} \begin{bmatrix} 0 & t_1 \end{bmatrix} dt_1 = \begin{bmatrix} t^2 / 6 & 0 \\ t^3 / 3 & 0 \end{bmatrix}
\]
\[
\phi(t, 0) = \begin{bmatrix} 1 + t^3 / 6 + ... & t + t^4 / 12 + ... \\ t^2 / 2 + t^5 / 30 + ... & 1 + t^3 / 3 + ...
\end{bmatrix}
\]

For controllability and observability the rank should be two.
where

\[
N_{k+1}(t) = N_k(t) A(t) + \frac{d}{dt} N_k(t)
\]

for \(k = 0, 1, 2, \ldots, n - 1\)

\[
N_n(t) = C(t)
\]

ILLUSTRATIVE EXAMPLES

Example 8.14. A system is given below, find the state and output equation of the system.

\[
\begin{align*}
\frac{y(s)}{u(s)} &= \frac{K}{(s+1)(s+2)(s^2+1)} \\
\end{align*}
\]

Solution:

\[
\begin{align*}
\frac{y(s)}{u(s)} &= G(s) = \frac{K}{(s+1)(s+2)(s^2+1)} \\
y(s) &= s^2y(s) + 3sy(t) + 3y(t) + 2y(t) = Ku(t) \\
\end{align*}
\]

Taking inverse Laplace:

\[
\begin{align*}
\dot{y}(t) + 3\dot{y}(t) + 3\ddot{y}(t) + 2\dddot{y}(t) &= Ku(t) \\
\end{align*}
\]

\[
\begin{align*}
\dot{y}(t) &= x_1 \\
\ddot{y}(t) &= x_2 \\
\dddot{y}(t) &= x_3 \\
\end{align*}
\]

\[
\begin{align*}
\dddot{y}(t) &= x_4 \\
\end{align*}
\]

\[
\begin{align*}
y(t) &= x_1 \\
y(t) &= [1 0 0 0] x(t)
\end{align*}
\]

Rewriting

\[
\begin{align*}
\frac{\dot{x}_1}{x_1} &= [0 1 0 0] x_1 + \frac{d}{dt} x_1 \\
\frac{\dot{x}_2}{x_2} &= [0 0 1 0] x_2 + \frac{d}{dt} x_2 \\
\frac{\dot{x}_3}{x_3} &= [0 0 0 1] x_3 + \frac{d}{dt} x_3 \\
\frac{\dot{x}_4}{x_4} &= [0 -2 -3 -3] x_4 + \frac{d}{dt} x_4 \\
\end{align*}
\]

Example 8.15. A system is described by the following transfer function

\[
G(s) = \frac{20(10s+1)}{s^3 + 3s^2 + 6s + 1}
\]

Find the state and output equation of the system.

Solution:

\[
\begin{align*}
\frac{y(s)}{u(s)} &= G(s) = \frac{20(10s+1)}{s^3 + 3s^2 + 6s + 1} \\
\frac{y(s)}{u(s)} &= \frac{X_1(s)}{u(s)} = \frac{20}{s^3 + 3s^2 + 6s + 1} \\
\end{align*}
\]

Consider

\[
\begin{align*}
\frac{X_1(s)}{u(s)} &= \frac{20}{s^3 + 3s^2 + 2s + 1} \\
X_1(s) &= 20 u(s)
\end{align*}
\]

Taking inverse Laplace:

\[
\begin{align*}
X_1(t) + 3X_2(t) + 2X_3(t) + X_4(t) &= 20 u(t)
\end{align*}
\]

Select

\[
\begin{align*}
X_1 &= X_2 \\
X_2 &= X_3 \\
X_3 &= X_4 \\
X_4 &= 20 u - 3X_1 - 2X_2 - X_3
\end{align*}
\]

Rewriting

\[
\begin{align*}
\begin{bmatrix}
\dot{X}_1 \\
\dot{X}_2 \\
\dot{X}_3 \\
\dot{X}_4
\end{bmatrix} &=
\begin{bmatrix}
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 \\
-2 & -3 & -3 & -3
\end{bmatrix}
\begin{bmatrix}
X_1 \\
X_2 \\
X_3 \\
X_4
\end{bmatrix} +
\begin{bmatrix}
0 \\
0 \\
0 \\
K
\end{bmatrix} u(t)
\end{align*}
\]

or,

\[
\begin{align*}
y(s) &= (10s + 1) X_1(s)
\end{align*}
\]

\[
\begin{align*}
y(t) &= X_1(t) + 10X_2(t)
\end{align*}
\]

\[
\begin{align*}
y(t) &= [1 10 0] X(t)
\end{align*}
\]
Example 8.16. A system characterised by the transfer function

\[ \frac{Y(s)}{u(s)} = \frac{2}{s^3 + 6s^2 + 11s + 6} \]

Find the state and output equation in matrix form and also test the controllability and observability of the system.

**Solution:** \((s^3 + 6s^2 + 11s + 6) Y(s) = 2u(s)\).

Taking inverse Laplace,

\[ Y(t) + 6Y(t) + 11Y(t) + 6Y(t) = 2u(t) \]

let

\[ Y(t) = x_1 \]
\[ \dot{Y}(t) = x_2 \]
\[ \ddot{Y}(t) = x_3 \]
\[ Y(t) = x_4 \]

\[ Y(t) = x_4 = x_3 = 2u(t) - 6x_3 - 11x_2 - 6x_1 \]

Rewriting

\[
\begin{bmatrix}
    \dot{x}_1 \\
    \dot{x}_2 \\
    \dot{x}_3 \\
    \dot{x}_4
\end{bmatrix}
= \begin{bmatrix}
    0 & 1 & 0 & 0 \\
    0 & 0 & 1 & 0 \\
    -6 & -11 & -6 & 2 \\
    2 & 0 & 0 & 1
\end{bmatrix}
\begin{bmatrix}
    x_1 \\
    x_2 \\
    x_3 \\
    x_4
\end{bmatrix}
+ \begin{bmatrix}
    0 \\
    0 \\
    0 \\
    0
\end{bmatrix} u(t)
\]

\[ y(t) = \begin{bmatrix}
    1 & 0 & 0 & 0
\end{bmatrix}
\begin{bmatrix}
    x_1 \\
    x_2 \\
    x_3 \\
    x_4
\end{bmatrix}
\]

For controllability

\[ Q = \begin{bmatrix}
    b : Ab : A^2 b
\end{bmatrix}
\]

\[ A = \begin{bmatrix}
    0 & 0 & 1 \\
    1 & 0 & 0 \\
    -6 & -11 & -6
\end{bmatrix}, \quad b = \begin{bmatrix}
    0 \\
    0 \\
    0
\end{bmatrix}
\]

\[ Ab = \begin{bmatrix}
    0 & 1 & 0 & 0 \\
    0 & 0 & 1 & 0 \\
    -6 & -11 & -6 & 2
\end{bmatrix}
\]

\[ A^2 b = \begin{bmatrix}
    2 \\
    50
\end{bmatrix}
\]

\[ Q = \begin{bmatrix}
    0 & 0 & 2 \\
    0 & 2 & -12 \\
    2 & -12 & 50
\end{bmatrix}
\]

\[ |Q| \neq 0 \]

It is full of rank, hence the system is controllable.

For observability:

\[ Q^T = \begin{bmatrix}
    1 & 0 & 0 \\
    0 & 0 & 1 \\
    0 & 1 & 0
\end{bmatrix}
\]

\[ A^T C^T = \begin{bmatrix}
    1 & 0 & 0 \\
    0 & 0 & 1 \\
    0 & 1 & 0
\end{bmatrix}
\]

\[ A^T C^T = \begin{bmatrix}
    1 & 0 & 0 \\
    0 & 0 & 1 \\
    0 & 1 & 0
\end{bmatrix}
\]

\[ Q' = \begin{bmatrix}
    1 & 0 & 0 \\
    0 & 0 & 1 \\
    0 & 1 & 0
\end{bmatrix}
\]

Rank of the matrix = 3, Hence the system is observable.

Example 8.17. A system is described by the following equations

\[ \dot{x}(t) = \begin{bmatrix}
    -1 & -1 & 1 \\
    0 & 0 & 0 \\
    1 & 1 & 1
\end{bmatrix} x(t) + \begin{bmatrix}
    0 & 0 & 1 \\
    1 & 1 & 0 \\
    1 & 1 & 0
\end{bmatrix} u(t) \]

\[ y(t) = \begin{bmatrix}
    1 & 2 \\
    1 & 0 \\
    1 & 1
\end{bmatrix} x(t) \]

Find the transfer function of the system.

**Solution:** The given system is MIMO system.

\[ G(s) = C[sI - A]^{-1} B \]

\[ sI - A = \begin{bmatrix}
    s & 0 & 0 \\
    0 & s & -1 \\
    0 & -2 & s + 1
\end{bmatrix} \]

\[ [sI - A]^{-1} = \begin{bmatrix}
    \frac{1}{s+2} & \frac{1}{s+3} \\
    -\frac{1}{s+2} & \frac{1}{s+3}
\end{bmatrix} \]

\[ C[sI - A]^{-1} B = \begin{bmatrix}
    1 & 0 & 0 \\
    0 & 1 & 1 \\
    0 & 1 & 1
\end{bmatrix} \begin{bmatrix}
    \frac{1}{s^2 + 3s + 2} \\
    \frac{1}{s^2 + 3s + 2} \\
    \frac{1}{s^2 + 3s + 2}
\end{bmatrix} \]

\[ G(s) = \frac{s^3 - 3s^2 - 9s + 2}{s^2 + 3s + 2} \]

is the required transfer function.
Example 8.18. A system is characterized by the equation

\[ \frac{y(s)}{u(s)} = \frac{20}{s^3 + 5s^2 + 8s + 2} \]

Find its state and output equation and express in matrix form.

Solution:

\[
\frac{y(s)}{u(s)} = \frac{X_1(s)}{u(s)} = \frac{20}{s^3 + 5s^2 + 8s + 2} \quad (4s + 2)
\]

\[
X_1(s) = \frac{20}{s^3 + 5s^2 + 8s + 2}u(s)
\]

Taking inverse laplace:

\[ x_1 + 5x_1 + 8x_1 + 2x_1 = 20u \]

Select:

\[
\begin{align*}
x_1' & = x_2 \\
x_2' & = x_3 \\
x_3' & = x_4
\end{align*}
\]

Rewriting:

\[
\begin{bmatrix}
x_1 \\
x_2 \\
x_3
\end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -2 & -5 & 20 \end{bmatrix} \begin{bmatrix}
x_1 \\
x_2 \\
x_3
\end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u
\]

\[
y(s) = \frac{X_1(s)}{s^3 + 3s^2 + 2s + 10} = 4s + 2
\]

Example 8.19. A system is described by the following transfer function

\[ G(s) = \frac{s + 2}{s^3 + 3s^2 + 2s + 10} \]

Find the values of A, b and \( C^T \)

Solution:

\[
\begin{align*}
\frac{Y(s)}{u(s)} & = \frac{s + 2}{s^3 + 3s^2 + 2s + 10} \\
\frac{Y(s)}{u(s)} & = \frac{X_1(s)}{u(s)} = \frac{1}{s^3 + 3s^2 + 2s + 10} (s + 2) \\
\frac{X_1(s)}{u(s)} & = \frac{1}{s^3 + 3s^2 + 2s + 10}
\end{align*}
\]
\[ AB = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -3 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = -2 \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = A^2B = \begin{bmatrix} 1 \\ 4 \\ 0 \end{bmatrix} \]

\[ Q = [B : AB : A^2B] = \begin{bmatrix} 1 & -1 & 1 \\ 1 & -2 & 4 \\ 0 & 0 & 0 \end{bmatrix} \]

Rank of the matrix = 2
Hence, the system is uncontrollable

For observability
\[ Q' = [C^T : A^T C^T : A^{2T} C^T] = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -3 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} = 0 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \]
\[ A^{T}C^T = \begin{bmatrix} 1 \\ 0 \\ 18 \end{bmatrix} \]
\[ A^{T}C^T = \begin{bmatrix} 1 \\ -1 \\ 1 \\ 0 \\ 0 \\ 2 \end{bmatrix} \cdot \begin{bmatrix} 18 \end{bmatrix} \]

Rank of matrix = 2
Hence, system is unobservable

The given system is uncontrollable and unobservable.

Example 8.21. Consider the system shown in fig. 8.9 and investigate whether it is observable or not.

Solution:
\[ \dot{X}_1 = -X_1 + f_1 \]
\[ \dot{X}_2 = -2X_2 + f_2 \]

Fig. 8.9.

For observability
\[ Q' = [C^T : A^T C^T] \]
\[ A = \begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix} \cdot \begin{bmatrix} 0 \\ 0 \\ -2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ -2 \end{bmatrix} \]
\[ C = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \cdot \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \]

\[ A^{T}C^T = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -3 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = -2 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \]

Example 8.22. A system is described by the matrices
\[ A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & -2 & -3 \end{bmatrix}, \quad b = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 2 & 0 \end{bmatrix} \]

Determine the transfer function.
Solution:
\[ sI - A = \begin{bmatrix} s & 0 & 0 \\ 0 & s & 0 \\ 0 & 0 & s \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -2 & -3 \end{bmatrix} = \begin{bmatrix} s^3 + 3s^2 + 2s + 3 \\ 0 \\ 0 \end{bmatrix} \]
\[ [sI - A]^{-1} = \begin{bmatrix} \frac{1}{s^3 + 3s^2 + 2s + 3} \\ 0 \\ 0 \end{bmatrix} \]
\[ C^T [sI - A]^{-1} b = \begin{bmatrix} s & 3s & 0 \\ 0 & s & s \\ 0 & 0 & s \end{bmatrix} \begin{bmatrix} \frac{1}{s^3 + 3s^2 + 2s + 3} \\ 0 \\ 0 \end{bmatrix} = \frac{2s + 1}{s^3 + 3s^2 + 2s + 3} \]
Hence required transfer function is

\[ \frac{2s+1}{s^3 + 3s^2 + 2s} \]

Example 8.23. Obtain the transfer function of the system.

![Circuit Diagram]

Fig. 8.10.

Solution: From fig. 8.10.

\[ \dot{x}_1 = -2x_1 - x_2 + 3u \]
\[ \dot{x}_2 = -3x_1 - 2x_2 + 4u \]

Output equation:

\[ y = 2x_1 + x_2 \]

\[ \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -2 & -1 \\ -3 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 3 \\ 4 \end{bmatrix} u \]

\[ y = \begin{bmatrix} 2 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \]

\[ [sI - A]^{-1} = \frac{1}{s^2 + 4s + 1} \begin{bmatrix} s + 2 & -1 \\ -3 & s + 2 \end{bmatrix} \]

\[ C^T [sI - A]^{-1} b = \frac{7s^3 + 3}{s^2 + 4s + 1} \]

Hence, required transfer function is

\[ \frac{7s^3 + 3}{s^2 + 4s + 1} \]

Example 8.24. Consider the following matrix

\[ \dot{x} = \begin{bmatrix} 0 & 1 \\ -6 & -5 \end{bmatrix} x(0) = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \]

Find the state transition matrix, also determine \( x(t) \)

Solution:

Given

\[ A = \begin{bmatrix} 0 & 1 \\ -6 & -5 \end{bmatrix} \]

\[ \Phi(s) = [sI - A]^{-1} = \frac{1}{s^2 + 5s + 6} \begin{bmatrix} s + 5 & 1 \\ -6 & s \end{bmatrix} \]

Example 8.25. Write the state equation for the circuit shown in fig. 8.11.

Solution: Select the state variables as \( i_1, i_2, \) and \( V_c \)

\( IV \) in mesh (1):

\[-24 \frac{di_1}{dt} - 1i_1 - V_1 + V_c = 0 \]

\( IV \) in mesh (2):

\[-12 \frac{di_2}{dt} - 2i_2 - V_2 + V_c = 0 \]

KCL at node \( x \):

\[ i_1 + i_2 + 2 \frac{dV_c}{dt} = 0 \]

Example 8.26. Write the state equation for the circuit shown in fig. 8.12.

\[ \begin{bmatrix} i_1 \\ i_2 \\ V_c \end{bmatrix} = \begin{bmatrix} -1/2 & 0 & 1/2 \\ 0 & -2 & 1 \\ -1/2 & -1/2 & 0 \end{bmatrix} \begin{bmatrix} i_1 \\ i_2 \\ V_c \end{bmatrix} + \begin{bmatrix} -1/2 \\ 0 \\ -1 \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \end{bmatrix} \]
Solution: Select the state variables as \( V_c \), \( V_{c1} \), \( V_{c2} \), and \( i_L \).

**KVL in mesh (1):**

\[
V_v - 1i_1 - V_{c1} = 0 \quad \text{(8.56)}
\]

**KVL in mesh (2):**

\[
V_{c1} - 1 \frac{di_1}{dt} - 1i_R = 0 \quad \text{(8.57)}
\]

**KVL in mesh (3):**

\[
i_R - V_{c2} = 0
\]

**KCL at node (a):**

\[
i_1 = i_{c1} + i_L
\]

or

\[
i_1 = 2 \frac{dV_{c1}}{dt} + i_L
\]

**KCL at node b:**

\[
i_L = i_R + i_{c2}
\]

\[
i_R = i_L - i_{c2} = i_L - 2 \frac{dV_{c2}}{dt}
\]

From equation (8.58) and (8.60):

\[
i_L - 2 \frac{dV_{c2}}{dt} - V_{c2} = 0
\]

\[
\frac{dV_{c2}}{dt} = \frac{1}{2} i_L - \frac{1}{2} V_{c2}
\]

\[
i_R = \frac{V_{c2}}{2} = V_{c3}
\]

From eqn(8.57):

\[
V_{c1} - \frac{di_2}{dt} - V_{c3} = 0
\]

\[
\frac{di_2}{dt} = V_{c1} - V_{c3}
\]

Put the value of \( i_L \) form 8.59 in eqn 8.56:

\[
-V_{c1} - i_L - 2 \frac{dV_{c2}}{dt} - V_v = 0
\]

\[
\frac{dV_{c1}}{dt} = \frac{1}{2} V_v - \frac{1}{2} i_L - \frac{1}{2} V_{c1}
\]

\[
\begin{bmatrix}
V_{c1} \\
V_{c2} \\
i_L
\end{bmatrix}
= \begin{bmatrix}
-1/2 & 0 & -1/2 \\
0 & -1/2 & 1/2 \\
1 & -1 & 0
\end{bmatrix}
\begin{bmatrix}
V_{c1} \\
V_{c2} \\
i_L
\end{bmatrix}
+ \begin{bmatrix}
1/2 \\
0 \\
0
\end{bmatrix}
\begin{bmatrix}
V_v \\
i_L
\end{bmatrix}
\]

**Example 8.27:** Write the state equation for the circuit shown in fig. 8.13.

**Solution:** Select the state variables as \( V_C \) and \( i_L \).

**KVL in mesh (1):**

\[
i(t) - 1 - V_C = 0
\]

**KVL in mesh (2):**

\[
-2i_L - 1 \frac{di_L}{dt} + V_C = 0
\]

**KVL in node a:**

\[
i(t) = 2 \frac{dV_C}{dt} + i_L
\]

\[
\frac{dV_C}{dt} = \frac{1}{2} i(t) - \frac{1}{2} i_L
\]

\[
\begin{bmatrix}
V_C \\
i_L
\end{bmatrix}
= \begin{bmatrix}
0 & -1/2 \\
1 & -2
\end{bmatrix}
\begin{bmatrix}
V_C \\
i_L
\end{bmatrix}
+ \begin{bmatrix}
1/2 \\
0
\end{bmatrix}
\begin{bmatrix}
i_L
\end{bmatrix}
\]

**Ans.**

**Example 8.28:** Express the system performance by a suitable state space representation of the system shown in fig. 8.15.

**Solution:** System equations

\[
\dot{\theta} + \beta \dot{\theta} = K_T i_L
\]

\[
\dot{\theta} = \frac{L_s i_L + R_s i_L}{V_m + K_r \dot{\theta}}
\]

Where

\[
\beta = \text{Coefficient of viscous friction}
\]

\[
K_T = \text{Torque constant of the motor}
\]

\[
K_r = \text{Back e.m.f. constant of the motor}
\]

\[
V_m = K_v V_{in}
\]

\[
V_{in} = u
\]

\[
V_m = K_u u
\]
Select state variables as
\[
\begin{aligned}
0 &= x_1 \\
\dot{x}_1 &= x_2 \\
\dot{x}_2 &= \frac{\dot{\theta}}{I} x_2 + \frac{K_r}{I} x_3 \\
L_\alpha \dot{x}_3 + R_\alpha x_3 &= V_a + K_v x_2 \\
\ddot{x}_3 &= \frac{R_\alpha}{L_\alpha} x_3 + \frac{K_v}{L_\alpha} x_2 + \frac{K_e}{L_\alpha} u
\end{aligned}
\]

or,
\[
\begin{aligned}
\dot{x}_1 &= x_2 \\
\dot{x}_2 &= x_3 \\
\dot{x}_3 &= \frac{K_r}{I} x_3 \\
\dot{x}_4 &= \frac{-R_\alpha}{L_\alpha} x_3 + \frac{K_v}{L_\alpha} x_2 + \frac{K_e}{L_\alpha} u
\end{aligned}
\]

Rewriting
\[
\begin{bmatrix}
\dot{x}_1 \\
\dot{x}_2 \\
\dot{x}_3 \\
\dot{x}_4
\end{bmatrix} =
\begin{bmatrix}
0 & 1 & 0 & 0 \\
0 & -\beta/I & K_r/I & 0 \\
0 & K_v/L_\alpha & -R_\alpha/L_\alpha & K_e/L_\alpha \\
0 & 0 & 0 & 1
\end{bmatrix}
\begin{bmatrix}
x_1 \\
x_2 \\
x_3 \\
x_4
\end{bmatrix} +
\begin{bmatrix}
0 \\
0 \\
K_e/L_\alpha \\
0
\end{bmatrix} u
\]

\[
y = [1 \ 0 \ 0] x
\]

Example 8.29. Construct the state model of a system characterized by the differential equation.
\[
\frac{d^3 y}{dt^3} + 6 \frac{d^2 y}{dt^2} + 11 \frac{dy}{dt} + 6y = u
\]

Give the block diagram representation of the state model.

Solution:
\[
\frac{d^3 y}{dt^3} + 6 \frac{d^2 y}{dt^2} + 11 \frac{dy}{dt} + 6y = u
\]

Take Laplace transform
\[
s^3 Y(s) + 6s^2 Y(s) + 11s Y(s) + 6Y(s) = u(s)
\]

\[
(s^3 + 6s^2 + 11s + 6) Y(t) = u(t)
\]

Taking inverse Laplace transform
\[
Y(t) + 6Y(t) + 11Y(t) + 6 Y(t) = u(t)
\]

let
\[
Y(t) = x_1 \\
\dot{Y}_1(t) = x_2 \\
\dot{Y}_2(t) = x_3 \\
\ddot{Y}_3(t) = x_4
\]

\[
Y(t) = \dot{x}_3 = 2u(t) - 6x_3 - 11x_2 - 6x_1
\]

\[
\begin{bmatrix}
\dot{x}_1 \\
\dot{x}_2 \\
\dot{x}_3 \\
\dot{x}_4
\end{bmatrix} =
\begin{bmatrix}
0 & 1 & 0 & 0 \\
0 & -\beta/I & K_r/I & 0 \\
0 & K_v/L_\alpha & -R_\alpha/L_\alpha & K_e/L_\alpha \\
0 & 0 & 0 & 1
\end{bmatrix}
\begin{bmatrix}
x_1 \\
x_2 \\
x_3 \\
x_4
\end{bmatrix} +
\begin{bmatrix}
0 \\
0 \\
K_e/L_\alpha \\
0
\end{bmatrix} u
\]

\[
y(t) = [1 \ 0 \ 0] x(t)
\]

Example 8.30. A linear time invariant system is characterized by the homogeneous state equation
\[
\begin{bmatrix}
\dot{x}_1 \\
\dot{x}_2
\end{bmatrix} =
\begin{bmatrix}
1 & 0 \\
1 & 1
\end{bmatrix} x(t)
\]

Compute the solution of homogeneous equation, assuming the initial state vector
\[
x(0) =
\begin{bmatrix}
1 \\
0
\end{bmatrix}
\]

Solution: Given
\[
A =
\begin{bmatrix}
1 & 0 \\
1 & 1
\end{bmatrix}
\]

\[
sI - A =
\begin{bmatrix}
s & 0 \\
0 & s
\end{bmatrix} -
\begin{bmatrix}
1 & 0 \\
1 & 1
\end{bmatrix} =
\begin{bmatrix}
s - 1 & 0 \\
1 & s - 1
\end{bmatrix}
\]

\[
(sI - A)^{-1} =
\begin{bmatrix}
\frac{1}{s - 1} & 0 \\
\frac{1}{(s - 1)^2} & \frac{1}{s - 1}
\end{bmatrix}
\]

\[
\phi(t) = e^t \begin{bmatrix}
e^t & 0 \\
e^t & e^t
\end{bmatrix}
\]

\[
x(t) = \phi(t) x(0) =
\begin{bmatrix}
e^t & 0 \\
e^t & e^t
\end{bmatrix}
\begin{bmatrix}
1 \\
0
\end{bmatrix} =
\begin{bmatrix}
e^t \\
e^t
\end{bmatrix}
\]

Ans.
8.12. SIMILARITY TRANSFORMATION

Similarity transformation means transformation of one set of dynamic equations to another set of dynamic equations. Consider the dynamic equation of a single input single output system:

\[
\frac{dx(t)}{dt} = Ax(t) + Bu(t) \\
y(t) = Cx(t) + Du(t) \\
x(t) = \text{state vector} (n \times 1) \\
u(t) = \text{input vector} \\
y(t) = \text{output vector}
\]  

Where

- \( \frac{dx(t)}{dt} = Ax(t) + Bu(t) \)  
- \( y(t) = Cx(t) + Du(t) \)  
- \( x(t) = \text{state vector} (n \times 1) \)  
- \( u(t) = \text{input vector} \)  
- \( y(t) = \text{output vector} \)

Let the dynamic equations (8.67) & (8.68) be transformed into another set of equations.  
let  
\( x(t) = P\bar{x}(t) \)  
or,  
\( \bar{x}(t) = P^{-1}x(t) \)  
Where, \( P = \text{non-singular matrix} (n \times n) \)  
then the transformed dynamic equations will be

\[
\frac{d\bar{x}(t)}{dt} = \bar{A}\bar{x}(t) + \bar{B}u(t) \\
\bar{y}(t) = \bar{C}\bar{x}(t) + \bar{D}u(t)
\]

Take the derivative of equation (8.70):

\[
\frac{d\bar{x}(t)}{dt} = P^{-1}dx(t) \\
\frac{dy(t)}{dt} = (P^{-1}C)x(t) + (P^{-1}D)u(t)
\]

Put the value of \( \frac{dx(t)}{dt} \) from equation (8.67) in equation (8.73):

\[
\frac{d\bar{x}(t)}{dt} = P^{-1}Ax(t) + P^{-1}Bu(t)
\]

Put the value of \( x(t) \) from (8.69) in equation (8.74):

\[
\frac{d\bar{x}(t)}{dt} = P^{-1}APx(t) + P^{-1}Bu(t)
\]

Compare equation (8.71) with (8.75):

- \( \bar{A} = P^{-1}AP \)
- \( \bar{B} = P^{-1}B \)

Put the value of \( x(t) \) from equation (8.69) in equation (8.68):

\[
y(t) = CP\bar{x}(t) + Du(t)
\]

Compare equation (8.76) with equation (8.72):

- \( \bar{C} = CP \)
- \( \bar{D} = D \)

By transformation, there is no change in characteristic equation, eigen values and transfer function.

8.13. STATE SPACE REPRESENTATION OF TRANSFER FUNCTION SYSTEMS:

8.13.0. State Space Representation in Canonical Form

Consider the transfer function

\[
Y(s) = \frac{b_0s^n + b_1s^{n-1} + \ldots + b_{n-1}s + b_n}{s^n + a_1s^{n-1} + \ldots + a_n}
\]

Let the dynamic equations (8.77) be written as

\[
\frac{Y(s)}{u(s)} = \frac{b_0s^n + b_1s^{n-1} + \ldots + b_{n-1}s + b_n}{s^n + a_1s^{n-1} + \ldots + a_n}
\]

Multiply both sides by \( u(s) \):

\[
Y(s) = b_0u(s) + \left[\frac{b_1}{s^n} + \frac{b_2}{s^{n-1}} + \ldots + \frac{b_n}{s}\right]u(s) + \ldots + \left[\frac{b_{n-1}}{s^n} + \frac{b_n}{s^{n-1}}\right]u(s) + \ldots + \left[\frac{b_n}{s^n}\right]u(s)
\]

Equation (8.81) can be written as

\[
\dot{\hat{y}}(s) = \frac{b_0}{s^n} + \frac{b_1}{s^{n-1}} + \ldots + \frac{b_n}{s^n} = Q(s)
\]

Where \( Q(s) \) is a dummy variable

Equate L.H.S of equation (8.82) to \( Q(s) \):

\[
\dot{\hat{y}}(s) = Q(s) = \left[\frac{b_1}{s^n} + \ldots + \frac{b_n}{s^n}\right] + (b_{n-1}a_n1) + (b_{n-2}a_n2) + \ldots + (b_1a_n) + (b_0a_n)
\]

Equate R.H.S of equation (8.82) to \( Q(s) \):

\[
u(s) = Q(s) + \left[\frac{s^n + a_1s^{n-1} + \ldots + a_n}{s^n} + a_n\right]
\]

Select the state variables

- \( X_1(s) = Q(s) \)
- \( X_2(s) = sQ(s) \)
- \( X_3(s) = s^2Q(s) \)
- \( \ldots \)
- \( X_{n-1}(s) = s^{n-2}Q(s) \)
- \( X_n(s) = s^{n-1}Q(s) \)
Multiplying both sides of first equation of equation (8.85) by s
\[ sX_1(s) = sQ(s) = X_n(s) \]
\[ sX_1(s) = X_2(s) \]
similarly
\[ s^2X_1(s) = s^2Q(s) = X_3(s) \]
\[ sX_2(s) = X_3(s) \]
\[ \vdots \]
\[ sX_{n-1}(s) = x_n(s) \]  \( (8.86) \)

Equation (8.86) can be written as
\[ \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_{n-1} \end{bmatrix} = \begin{bmatrix} x_2 \\ x_3 \\ \vdots \\ x_n \end{bmatrix} \]  \( (8.87) \)

Also,
\[ S^n Q(s) = sX_n(s) \]
Equation (8.84) can be written as (with the help of equation 8.85)
\[ sX_n(s) = u(s) + [-a_1 X_n(s) \ldots -a_{n-1} X_2(s) - a_n X_1(s)] \]
or,
\[ \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} x_2 \\ x_3 \\ \vdots \\ x_n \end{bmatrix} + u(s) \]  \( (8.88) \)

Equation (8.87) & (8.88) can be written in matrix form
\[ \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_{n-1} \end{bmatrix} = \begin{bmatrix} 0 & 0 & \cdots & 0 \\ 1 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ -a_2 & -a_3 & \cdots & -a_{n-1} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_{n-1} \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix} u(s) \]  \( (8.89) \)

From equation (8.80) and (8.83)
\[ Y(s) = b_n u(s) + (b_{n-1} - a_n b_n) s^{n-1} Q(s) + \ldots + (b_1 - a_n b_n) s Q(s) + (b_0 - a_n b_n) Q(s) \]
from equation (8.85)
\[ Y(s) = b_n u(s) + (b_{n-1} - a_n b_n) x_n + \ldots + (b_1 - a_n b_n) X_2(s) + (b_0 - a_n b_n) X_1(s) \]
Taking inverse laplace
\[ y = b_n u(s) + (b_{n-1} - a_n b_n) x_n + \ldots + (b_1 - a_n b_n) X_2(s) + (b_0 - a_n b_n) X_1(s) \]
or,
\[ y = \left[ (b_n - a_n b_n), (b_{n-1} - a_n b_n), \ldots, (b_1 - a_n b_n) \right] \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} + b_n u(s) \]  \( (8.90) \)

Equation (8.89) and (8.90) are said to be in the controllable canonical form (C).

Now compare the equation (8.89) with (8.71)

| \overrightarrow{A} | = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ -a_n & -a_{n-1} & -a_{n-2} & \cdots & -a_1 \end{bmatrix} = P^{-1}AP \]  \( (8.91) \)

\[ \overrightarrow{B} = P^{-1}B = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix} \]  \( (8.92) \)

\[ P = SM \]  \( (8.93) \)

\[ S = [B \ AB \ A^2 \ B \ \ldots \ A^{n-2} \ B] \]  \( (8.94) \)

\[ M = \begin{bmatrix} a_{n-1} & a_{n-2} & \cdots & a_1 & 1 \\ a_{n-2} & \vdots & \ddots & \vdots & \vdots \\ \vdots & \vdots & \ddots & 1 & \vdots \\ a_1 & 1 & \cdots & 0 & 0 \\ 1 & 0 & \cdots & 0 & 0 \end{bmatrix} \]  \( (8.95) \)

\[ \text{The characteristic equation is} \]
\[ |sI - \overrightarrow{A}| = s^n + a_1 s^{n-1} + \ldots + a_{n-1} s + a_n \]

L13.02 Observable Canonical Form
Consider the transfer function given in equation (8.77) Equation (8.77) can be written as
\[ Y(s) = \left[ \begin{bmatrix} a_{n-1} s^{n-1} + \ldots + a_1 s + a_0 \right] u(s) + b_0 s^{n-2} + \ldots + b_{n-1} s + b_n \right] Y(s) \]  \( \text{(8.96)} \)
or,
\[ s^n Y(s) + a_1 s^{n-1} Y(s) + \ldots + a_{n-1} s Y(s) + a_n Y(s) = b_0 s^{n-2} u(s) + \ldots + b_{n-1} s u(s) + b_n u(s) \]  \( \text{(8.97)} \)

Rearrange equation (8.97) by transferring RHS to LHS
\[ s^n Y(s) - b_0 s^{n-1} Y(s) - b_1 s^{n-2} Y(s) - \ldots - b_{n-1} s Y(s) - b_n u(s) = 0 \]
or, \[ s^n [Y(s) - b_n Y(s)] + s^{n-1} [a_n Y(s) - b_1 u(s)] + s^{n-2} [a_2 Y(s) - b_2 u(s)] + \ldots + s[n-1] Y(s) - b_{n-1} u(s) + a_n Y(s) - b_n u(s) = 0 \]

Divide the equation (8.104) by \( s_n \)

\[ [Y(s) - b_n Y(s)] + s^{n-1} [a_1 Y(s) - b_1 u(s)] + s^{n-2} [a_2 Y(s) - b_2 u(s)] + \ldots + s[n-1] Y(s) - b_{n-1} Y(s) - b_n u(s) = 0 \]

Rearrange equation (8.99)

\[ Y(s) = b_n u(s) + s^{n-1} [b_1 u(s) - a_1 Y(s)] + s^{n-2} [b_2 u(s) - a_2 Y(s)] + \ldots + s[n-1] Y(s) - b_{n-1} Y(s) - b_n u(s) = 0 \]

Select the state variable as

\[
\begin{align*}
X_n(s) &= s^n Y(s) - \frac{1}{s} [b_1 u(s) - a_1 Y(s) + X_{n-1}(s)] \\
X_{n-1}(s) &= s^{n-1} Y(s) - \frac{1}{s} [b_2 u(s) - a_2 Y(s) + X_{n-2}(s)] \\
&\vdots \\
X_1(s) &= s^1 Y(s) - \frac{1}{s} [b_{n-1} u(s) - a_{n-1} Y(s) + X_1(s)] \\
X_0(s) &= s^0 Y(s) - \frac{1}{s} [b_n u(s) - a_n Y(s)]
\end{align*}
\]

Equation (8.101) can be written as

\[
\begin{align*}
sX_{n-1} &= [b_1 u(s) - a_1 Y(s) + X_{n-1}(s)] \\
sX_{n-2} &= [b_2 u(s) - a_2 Y(s) + X_{n-2}(s)] \\
&\vdots \\
sX_n &= [b_1 u(s) - a_1 Y(s) + X_1(s)] \\
sX_1 &= [b_n u(s) - a_n Y(s)]
\end{align*}
\]

\[ Y(s) = b_n u(s) + \frac{1}{s} [b_1 u(s) - a_1 Y(s) + X_0(s)] \]

Put equation (8.103) in equation (8.102)

\[
\begin{align*}
sX_n &= X_{n-1} - a_1 X_n + (b_1 - a_1 b_n) u(s) \\
sX_{n-1} &= X_{n-2} - a_2 X_n + (b_2 - a_2 b_n) u(s) \\
&\vdots \\
sX_2 &= X_1 - a_n X_n + (b_n - a_n b_n) u(s) \\
\end{align*}
\]

Taking inverse laplace of equation (8.104)

\[
\begin{align*}
x_n &= x_{n-1} - a_1 x_n + (b_1 - a_1 b_n) u \\
x_{n-1} &= x_{n-2} - a_2 x_n + (b_2 - a_2 b_n) u \\
&\vdots \\
x_1 &= x_1 - a_n x_n + (b_n - a_n b_n) u \\
x &= x_0 \end{align*}
\]

Rearrange the equation (8.105)

\[
\begin{align*}
x_1 &= -a_n x_n + (b_n - a_n b_n) u \\
x_2 &= x_1 - a_1 x_n + (b_1 - a_1 b_n) u \\
&\vdots \\
x_n &= x_{n-2} - a_2 x_n + (b_2 - a_2 b_n) u \\
\end{align*}
\]

Inverse laplace of equation (8.103)

\[
\begin{align*}
y &= b_n u(s) + x_n \\
&\vdots \\
&y = x_n + b_n u(s)
\end{align*}
\]

Equation (8.106) and (8.107) can be written in matrix form as

\[
\begin{bmatrix}
x_1 \\
x_2 \\
&\vdots \\
x_n \\
\end{bmatrix} = \begin{bmatrix}
0 & 0 & \ldots & 0 & -a_n \\
1 & 0 & \ldots & 0 & -a_{n-1} \\
&\vdots & \ddots & \vdots & \vdots \\
0 & 0 & \ldots & 1 & a_n \\
&\vdots & \ddots & \vdots & \vdots \\
\end{bmatrix} \begin{bmatrix}
x_1 \\
x_2 \\
&\vdots \\
x_n \\
\end{bmatrix} + \begin{bmatrix}
b_n - a_n b_n \\
b_{n-1} - a_{n-1} b_n \\
&\vdots & \ddots & \vdots \\
b_1 - a_1 b_n \\
&\vdots & \ddots & \ddots & \vdots \\
\end{bmatrix} u
\]

Equations (8.108) & (8.109) are said to be in the observable canonical form (OCF)

Compare equation (8.108) with equation (8.71)

\[
\bar{A} = \begin{bmatrix}
0 & 0 & \ldots & 0 & -a_n \\
1 & 0 & \ldots & 0 & -a_{n-1} \\
&\vdots & \ddots & \vdots & \vdots \\
0 & 0 & \ldots & 1 & a_n \\
&\vdots & \ddots & \ddots & \ddots \\
\end{bmatrix} \nonumber \tag{8.110}
\]

\[
\bar{B} = \begin{bmatrix}
b_n - a_n b_n \\
b_{n-1} - a_{n-1} b_n \\
&\vdots & \ddots & \ddots \\
b_1 - a_1 b_n \\
&\vdots & \ddots & \ddots & \ddots \\
\end{bmatrix} \nonumber \tag{8.111}
\]

Here, we use \( Q \) instead of \( P \) as used in controllable canonical form.

\[
\bar{A} = Q^{-1} AQ \nonumber \tag{8.112}
\]

\[
\bar{C} = CQ \nonumber \tag{8.113}
\]

where

\[
Q = [M^T V]^{-1} \nonumber \tag{8.114}
\]
8.13.0.3. Diagonal Canonical Form

Let the equation (8.77) be written as

\[ \frac{Y(s)}{u(s)} = b_0 + \frac{d_1}{s + \lambda_1} + \frac{d_2}{s + \lambda_2} + \ldots + \frac{d_n}{s + \lambda_n} \]

or,

\[ Y(s) = b_0 u(s) + \frac{d_1}{s + \lambda_1} u(s) + \frac{d_2}{s + \lambda_2} u(s) + \ldots + \frac{d_n}{s + \lambda_n} u(s) \]

Define the state variable as

\[ X_1(s) = \frac{1}{s + \lambda_1} u(s) \]
\[ X_2(s) = \frac{1}{s + \lambda_2} u(s) \]
\[ \vdots \]
\[ X_n(s) = \frac{1}{s + \lambda_n} u(s) \]

Above equations can be written as

\[ sX_1(s) = -\lambda_1 X_1(s) + u(s) \]
\[ sX_2(s) = -\lambda_2 X_2(s) + u(s) \]
\[ \vdots \]
\[ sX_n(s) = -\lambda_n X_n(s) + u(s) \]

The inverse laplace of (8.77.3)

\[ x_1 = -\lambda_1 x_1 + u \]
\[ x_2 = -\lambda_2 x_2 + u \]
\[ \vdots \]
\[ x_n = -\lambda_n x_n + u \]

Equations (8.117.7) & (8.117.8) are said to be in diagonal canonical form.

(Notes:
\[ \lambda_1, \lambda_2, \ldots \] roots of characteristic equation or eigen value of matrix A.
\[ g = \text{only 1, 1, 1, \ldots no other values (means it is controllable)} \]
If any of \( d_1, d_2, \ldots \) is absent then system is unobservable)

We can also select the state variables as

\[ X_1(s) = \frac{d_1}{s + \lambda_1} u(s) \]
\[ X_2(s) = \frac{d_2}{s + \lambda_2} u(s) \]
\[ \vdots \]
\[ X_n(s) = \frac{d_n}{s + \lambda_n} u(s) \]

Equation (8.117.9) can be written as

\[ sX_1(s) = -\lambda_1 X_1(s) + d_1 u(s) \]
\[ sX_2(s) = -\lambda_2 X_2(s) + d_2 u(s) \]
\[ \vdots \]

Laplace transform of (8.117.10)

\[ \dot{x}_1 = -\lambda_1 x_1 + d_1 u \]
\[ \dot{x}_2 = -\lambda_2 x_2 + d_2 u \]
\[ \vdots \]

from equation (8.117.1)

\[ Y(s) = b_0 u(s) + X_1(s) + X_2(s) + X_3(s) + \ldots X_n \]
Inverse laplace of (8.117.12)
\[ y = b_0 u + x_1 + x_2 + x_3 + \ldots + x_n \]
Equation (8.117.12) & (8.117.13) can be written in matrix form
\[
\begin{bmatrix}
\dot{x}_1 \\
\dot{x}_2 \\
\dot{x}_3 \\
\vdots \\
x_n
\end{bmatrix} =
\begin{bmatrix}
-\lambda_1 & \cdots & 0 & \cdots & 0 \\
0 & -\lambda_2 & \cdots & 0 & \cdots & 0 \\
0 & 0 & -\lambda_3 & \cdots & 0 & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
0 & 0 & 0 & \cdots & -\lambda_n
\end{bmatrix}
\begin{bmatrix}
x_1 \\
x_2 \\
x_3 \\
\vdots \\
x_n
\end{bmatrix} +
\begin{bmatrix}
d_1 \\
d_2 \\
d_3 \\
\vdots \\
d_n
\end{bmatrix} u
\]
\[ y = [1 1 1 \ldots 1] x_3 + b_0 u \]

Equation (8.117.14) & (8.117.15) are also called diagonal canonical form. This is an alternative approach.
let \[ x(t) = T \tilde{x}(t) \]
or, \[ \tilde{x}(t) = T^{-1} x(t) \]
derivative of equation (8.117.17)
\[ \frac{d\tilde{x}(t)}{dt} = T^{-1} dx(t) \]
Put the value of \[ \frac{dx(t)}{dt} \] from equation (8.67)
\[ \frac{d\tilde{x}(t)}{dt} = T^{-1} Ax(t) + T^{-1} B u(t) \]
or,
\[ \frac{d\tilde{x}(t)}{dt} = T^{-1} AT \tilde{x}(t) + T^{-1} B u(t) \]
compare equation (8.117.19) with (8.71)
\[ \tilde{A} = T^{-1} AT \]
\[ \tilde{B} = T^{-1} B \]
from equation (8.68) and (8.72)
\[ y(t) = CT \tilde{x}(t) + Du(t) \]
Compare equation (8.117.21) with (8.72)
\[ \tilde{C} = CT \]
and \[ \tilde{D} = D \]
\( T \) matrix can be formed by the use of eigen vectors of \( A \).
\[ T = [p_1 \ p_2 \ p_3 \ \ldots \ p_n] \]
If the matrix \( A \) is of CCF and \( A \) has distinct eigen values, the DCF transformation matrix is the vandermonde matrix, then
\[
\begin{bmatrix}
1 & 1 & \ldots & 1 \\
\lambda_1 & \lambda_2 & \ldots & \lambda_n \\
\lambda_1^2 & \lambda_2^2 & \ldots & \lambda_n^2 \\
\vdots & \vdots & \ddots & \vdots \\
\lambda_1^{n-1} & \lambda_2^{n-1} & \ldots & \lambda_n^{n-1}
\end{bmatrix}
\]
where, \( \lambda_1, \lambda_2, \lambda_3, \ldots, \lambda_n \) are the eigen values of \( A \).

Example 8.31. Consider the state space representation given in example 8.16, change it into diagonal canonical form (DCF).
Solution: Compare state equation (8.67) & (8.68)
\[ A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -6 & -11 & -6 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix}, \quad C = [1 \ 0 \ 0] \]
First calculate eigen vector of \( A \).
\[ |\lambda I - A| = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \lambda^3 + 6\lambda^2 + 11\lambda + 6 \]
or
\[ |\lambda I - A| = (\lambda + 1) (\lambda + 2) (\lambda + 3) = 0 \]
\[ \lambda_1 = -1, \lambda_2 = -2, \lambda_3 = -3 \]
Now, three eigenvalues are distinct, then we calculate \( \tilde{A}, \tilde{B} \) by the use of \( T \), where
\[ T = \begin{bmatrix} 1 & 1 & 1 \\ \lambda_1 & \lambda_2 & \lambda_3 \\ \lambda_1^2 & \lambda_2^2 & \lambda_3^2 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ -1 & -2 & -3 \\ 1 & 4 & 9 \end{bmatrix} \]
Note: students are advised to calculate $T^{-1}$ themselves

$$A = T^{-1}AT$$

$$= \begin{bmatrix}
3 & 2.5 & 0.5 \\
-3 & -4 & -1 \\
1 & 1.5 & 0.5
\end{bmatrix}$$

$$= \begin{bmatrix}
3 & 2.5 & 0.5 \\
-3 & -4 & -1 \\
1 & 1.5 & 0.5
\end{bmatrix} \begin{bmatrix}
1 & 1 & 1 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{bmatrix}$$

$$= \begin{bmatrix}
1 & 1 & 1 \\
-6 & -11 & -6 \\
1 & 4 & 9
\end{bmatrix}$$

$$B = T^{-1}B$$

$$= \begin{bmatrix}
3 & 2.5 & 0.5 \\
-3 & -4 & -1 \\
1 & 1.5 & 0.5
\end{bmatrix} \begin{bmatrix}
0 & 1 \\
0 & -2 \\
0 & -3
\end{bmatrix}$$

$$= \begin{bmatrix}
1 \\
-2 \\
1 \\
4 \\
9
\end{bmatrix}$$

$$C = CT = \begin{bmatrix}
1 & 0 & 0 \\
-1 & -2 & -3 \\
1 & 4 & 9
\end{bmatrix} \begin{bmatrix}
1 & 1 & 1 \\
1 & 0 & 0 \\
0 & 1 & 1
\end{bmatrix}$$

Now the state equation can be written in transform form (DCF) as with the help of equations (8.117.19) & (8.117.21)

$$\begin{bmatrix}
\dot{x}_1 \\
\dot{x}_2 \\
\dot{x}_3
\end{bmatrix} = \begin{bmatrix}
-1 & 0 & 0 \\
0 & -2 & 0 \\
0 & 0 & -3
\end{bmatrix} \begin{bmatrix}
x_1 \\
x_2 \\
x_3
\end{bmatrix} + \begin{bmatrix}
1 \\
-2 \\
1
\end{bmatrix} u$$

$$y = \begin{bmatrix}
1 & 1 & 1
\end{bmatrix} \begin{bmatrix}
x_1 \\
x_2 \\
x_3
\end{bmatrix}$$

From equations (8.117.23) & (8.117.24) it is clear that transformation matrix $T$ modifies the coefficient matrix into diagonal matrix. The diagonal elements of equation (8.117.23) are same as eigenvalues of $A$.

8.14. DECOMPOSITION TRANSFER FUNCTION

The process by which transfer function changes to state diagram or state equations is called decomposition of the transfer function. The transfer function can be decomposed by three methods known as direct decomposition, cascade decomposition and parallel decomposition.

8.14.1. Direct Decomposition

8.14.1.1 Direct decomposition to controllable canonical form (CCF)

Direct decomposition method is used when the transfer function of a system is not in factored form.

$$C(s) = \frac{b_0 s^n + b_{n-1} s^{n-1} + \ldots + b_1 s + b_0}{s^n + a_{n-1} s^{n-1} + \ldots + a_1 s + a_0}$$

The transfer function is

In the following steps are involved

1. Express the numerator & denominator in the form of rational polynomials.
2. Divide the numerator and denominator by the highest power of $s$ in the denominator or the numerator.
3. Multiply the numerator and denominator by $Q(s)$, where $Q(s)$ is a dummy variable.
4. Express the numerator and denominator separately and use the denominator equation to write an expression for $Q(s)$.
5. Draw the state variable diagram (SVD) for $Q(s)$ obtained in step 4.
8. Write expressions for the derivatives of state variables.
9. Feedback paths are determined by system poles and forward paths by system zero.

Example 8.32: Consider the following transfer function

$$\frac{C(s)}{R(s)} = \frac{s + 6}{s^2 + 5s + 6}$$

Find the state space representation.

Solution:

$$\frac{C(s)}{R(s)} = \frac{s + 6}{s^2 + 5s + 6}$$

Step 1: Divide numerator and denominator by $s^2$

$$\frac{C(s)}{R(s)} = \frac{s + 6}{s^2 + 5s + 6}$$

Step 2: Multiply numerator and denominator by $Q(s)$

$$\frac{C(s)}{R(s)} = \frac{s + 6}{1 + 5s^{-1} + 6s^{-2}}$$

Step 3: Express numerator and denominator separately.

$$C(s) = Q(s) [s^{-1} + 6s^{-2}]$$

$$R(s) = Q(s) [1 + 5s^{-1} + 6s^{-2}]$$

Equation (8.32.3) can be written as

$$R(s) = Q(s) + Q(s) [5s^{-1} + 6s^{-2}]$$

$$Q(s) = R(s) - Q(s) [5s^{-1} + 6s^{-2}]$$

Step 4: Draw SVD for equation (8.119(e))
Step 5: Draw SVD for equation (8.119(c)).

Fig. 8.20. CCF state diagram of transfer function

From diagram
\[ \begin{align*}
\dot{x}_1 &= x_2 \\
\dot{x}_2 &= r(t) - 6x_1 - 5x_2
\end{align*} \]

Therefore
\[ \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -6 & -5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} r(t) \]
\[ c(t) = 6x_1 + x_2 \]
\[ c(t) = [6 \\ 1] \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \]

Equations (8.119(f)) & (8.119(g)) are required state space representation.

The above equations are controllable canonical form (CCF).

8.14.1.2. Direct Decomposition to Observable Canonical Form (OCF)

Multiply numerator and denominator by \( s^n \) of equation (8.77)

\[ \frac{Y(s)}{U(s)} = \frac{b_2 + b_1 s^{-1} + \ldots + b_{n-1} s^{1-n} + b_n s^n}{1 + a_n s^{-1} + \ldots + a_{n-1} s^{1-n} + a_n s^n} \]

or,

\[ Y(s) = (b_2 + b_1 s^{-1} + \ldots + b_{n-1} s^{1-n} + b_n s^n) u(s) \]

\[ Y(s) = (a_1 s^{-1} + \ldots + a_{n-1} s^{1-n} + a_n s^n) u(s) \]

The output of integrators as designated as the state variables. The state variables are assigned in descending order from right to left. Apply SFG gain formula to state variable diagram and write the dynamic equation.

\[ \overline{A} = \begin{bmatrix} 0 & 0 & \ldots & 0 & -a_n \\ 1 & 0 & \ldots & 0 & -a_{n-1} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \ldots & 1 & a_n \end{bmatrix} \]

Consider the following example

Example 8.33. Obtain the state model of the system whose transfer function is given by

\[ T(s) = \frac{s^2 + 3s + 3}{s^3 + 2s^2 + 3s + 1} \]

(Solution)

\[ \frac{Y(s)}{U(s)} = \frac{s^2 + 3s + 3}{s^3 + 2s^2 + 3s + 1} \]

or,

\( Y(s) = (s^2 + 3s + 3) u(s) = (1 + 2s^{-1} + 3s^{-2} + s^{-3}) Y(s) \)

or,

\[ Y(s) = (s^2 + 3s + 3) u(s) - (2s^{-1} + 3s^{-2} + s^{-3}) Y(s) \]

with the help of equation (8.122) draw state variable diagram (SVD)

Fig. 8.21. OCF state diagram of the transfer function.

From OCF diagram shown in fig. 8.22.

\[ y = x_3 \]

\[ \begin{align*}
\dot{x}_1 &= u + x_2 - 2x_3 = 0x_1 + x_2 - 2x_3 + u \\
\dot{x}_2 &= 3u - 3x_3 + x_1 = 1x_1 + 0x_2 - 3x_3 + 3u \\
\dot{x}_3 &= 3u + x_1 = 0x_1 + 0x_2 + x_1 + 3u
\end{align*} \]

state equations are

\[ \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & -3 \\ 0 & 1 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix} u \]

\[ y = [0 \\ 0] \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \]

The above equations are required answer.
8.14.2. Cascade Decomposition

When the transfer function is in factored form then cascade decomposition is used. First make the transfer function in factored form, then draw the transfer function of each part and combine all parts by unity transmittances inserted between each SVD. Consider the following example.

Example 8.34. Consider the transfer function given in example 8.32 draw SVD and obtain state space model by series (cascade) decomposition.

Solution: The given transfer function can be written in factored form

\[
\frac{Y(s)}{U(s)} = \frac{s+6}{s+3(s+2)}
\]

Let,

\[
G_1(s) = \frac{s+6}{s+3} \quad \text{and} \quad G_2 = \frac{1}{s+2}
\]

First draw SVD for \( G_1 \)

\[
\begin{align*}
Y(s) &= 1+6s^{-1} Q(s) \\
U(s) &= 1+3s^{-1} Q(s) \\
Y(s) &= (1+6s^{-1}) Q(s) \\
U(s) &= (1+3s^{-1}) Q(s) \\
Q(s) &= u(s) - 3s^{-1} Q(s)
\end{align*}
\]

or,

SVD for \( G_1 \)

![Diagram of SVD for G1](image)

Now draw SVD for \( G_2 \)

\[
\begin{align*}
Y(s) &= 1 + 6s^{-1} Q(s) \\
U(s) &= 1 + 3s^{-1} Q(s) \\
Y(s) &= s^{-1} Q(s) \\
U(s) &= (1 + 2s^{-1}) Q(s) \\
Q(s) &= u(s) - 2s^{-1} Q(s)
\end{align*}
\]

or,

Draw SVD for \( G_2 \)

![Diagram of SVD for G2](image)

Combine SVD for \( G_1 \) & \( G_2 \) with unity transmittance, we get complete SVD

![Diagram of complete SVD cascade decomposition](image)

From SVD:

\[
\begin{align*}
\dot{x}_1 &= -2x_1 + x_2 + 6x_2 \\
\dot{x}_2 &= -3x_2 + u
\end{align*}
\]

Put the value of \( x_2 \) from (8.125) in (8.124), and simplified

\[
\begin{align*}
\dot{x}_1 &= -2x_1 + 3x_2 + u \\
\dot{x}_2 &= -3x_2 + u
\end{align*}
\]

In matrix form

\[
\begin{bmatrix}
\dot{x}_1 \\
\dot{x}_2
\end{bmatrix} =
\begin{bmatrix}
-2 & 3 \\
0 & -3
\end{bmatrix}
\begin{bmatrix}
x_1 \\
x_2
\end{bmatrix}
+ \begin{bmatrix}
u
\end{bmatrix}
\]

\[
y = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}
\]

The above equations are the required state space equations.

8.14.3. Parallel Composition

The method can be used when denominator of transfer function is in factored form and numerator may or may not be factored form. Transfer function is expanded by partial fraction and draw SVD for each part, connect them in parallel.

Example 8.35. Obtain the state space representation of the transfer function given in example 8.32

Solution:

\[
\frac{C(s)}{R(s)} = \frac{s+6}{(s+3)(s+2)}
\]

After partial fraction (8.126) can be written as

\[
\frac{C(s)}{R(s)} = \frac{-3}{s+3} + \frac{4}{s+2}
\]

(Students are advised do the partial fraction)

Now draw SVD for each term and then connect in parallel. The resultant SVD is shown in figure 8.23.

![Diagram of SVD parallel decomposition](image)

The set of state equations

\[
\begin{align*}
\dot{x}_1 &= -3x_1 + r \\
\dot{x}_2 &= -2x_2 + r
\end{align*}
\]

\[
\begin{bmatrix}
\dot{x}_1 \\
\dot{x}_2
\end{bmatrix} =
\begin{bmatrix}
-3 & 0 \\
0 & -2
\end{bmatrix}
\begin{bmatrix}
x_1 \\
x_2
\end{bmatrix}
+ \begin{bmatrix}
r
\end{bmatrix}
\]

\[
y = \begin{bmatrix} 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}
\]
Example 8.36. Given the dynamic equation
\[ \frac{dx(t)}{dt} = Ax(t) + Bu(t) \]
\[ y(t) = Cx(t) \]
where,
\[ A = \begin{bmatrix} -2 & 1 & 0 \\ 0 & -2 & 0 \\ -1 & -2 & -3 \end{bmatrix}, \quad B = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad C = [1 \ 0 \ 0] \]

Find the transformation matrix \( x(t) = P \tilde{x}(t) \).

Solution: First find out the characteristic equation
\[ |S - A| = s^3 + 7s^2 + 15s + 12 = 0 \]

Here,
\[ a_0 = 12, a_1 = 16, a_2 = 7, a_3 = 1 \]
\[ M = \begin{bmatrix} a_1 & a_2 & 1 \\ a_2 & a_3 & 0 \\ 1 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 16 & 7 & 1 \\ 7 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \]
\[ AB = \begin{bmatrix} -2 & 0 & 1 \\ 0 & -2 & 0 \\ -1 & -2 & -3 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ -6 \end{bmatrix} \]
\[ A^2B = \begin{bmatrix} -2 & 0 & 1 \\ 0 & -2 & 0 \\ -1 & -2 & -3 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ -6 \end{bmatrix} = \begin{bmatrix} 0 \\ 4 \\ -6 \end{bmatrix} \]
\[ S = [B \ AB \ A^2B] = \begin{bmatrix} 1 & -1 & 0 \\ 1 & -2 & 4 \\ 1 & -6 & 23 \end{bmatrix} \]
\[ S = \begin{bmatrix} 1 & -1 & 0 \\ 1 & -2 & 4 \\ 1 & -6 & 23 \end{bmatrix} \]
\[ \therefore \quad \text{Transformation matrix } P = SM \]
\[ P = \begin{bmatrix} 1 & -1 & 0 \\ 1 & -2 & 4 \\ 1 & -6 & 23 \end{bmatrix} = \begin{bmatrix} 16 & 7 & 1 \\ 6 & 5 & 1 \\ -3 & 1 & 1 \end{bmatrix} \quad \text{Ans.} \]

Example 8.37. Draw state variable diagram (SVD) for the given transfer function
\[ C(s) = \frac{5s}{3s^3 + 3s^2 + 1} \]
\[ R(s) = \frac{3}{3s^2 + 3s + 1} \]

Solution: Multiply numerator and denominator by \( s^2 \)
\[ \frac{C(s)}{R(s)} = \frac{\frac{5s}{s^2}}{\frac{3s^2}{s^2} + \frac{3s}{s^2} + \frac{1}{s^2}} \]
\[ \frac{C(s)}{R(s)} = \frac{5s}{3s^3 + 3s^2 + 1} Q(s) \]
\[ Q(s) = \frac{1}{3} R(s) - s Q(s) - \frac{1}{3} s^2 Q(s) \]
\[ Q(s) = \frac{1}{3} R(s) - Q(s) \left[ s^2 + \frac{1}{3} s^2 \right] \]

\[ R(s) = \frac{3Q(s)}{3s^3 + 3s^2 + 1} Q(s) \]

Example 8.38. Write the state equations
\[ \frac{d^2 c}{dt^2} + 6 \frac{dc}{dt} + 5c(t) = r(t) \]

Also draw the state variable diagram

Solution: Take laplace transform of given equation
\[ s^2C(s) + 6sC(s) + 5C(s) = R(s) \]
\[ \frac{C(s)}{R(s)} = \frac{1}{s^2 + 6s + 5} \]

Divide numerator and denominator by \( s^2 \)
\[ \frac{C(s)}{R(s)} = \frac{s^3}{s^3 + 6s^2 + 5s^2} Q(s) \]
\[ Q(s) = \frac{1}{s^3 + 6s^2 + 5s^2} Q(s) \]
\[ R(s) = Q(s) \left[ s^3 + 6s^2 + 5s^2 \right] \]
Effect of Pole-Zero Cancellation in Transfer Function

Consider the transfer function

\[ \frac{Y(s)}{u(s)} = \frac{b_n s^n + b_{n-1} s^{n-1} + \ldots + b_1 s + b_0}{s^n + a_{n-1} s^{n-1} + \ldots + a_1 s + a_0} \]

The above transfer function can be written in factored form

\[ \frac{Y(s)}{u(s)} = \frac{k(s - z_1)(s - z_2) \ldots (s - z_m)}{(s - p_1)(s - p_2) \ldots (s - p_k)} \]

\[ = \sum_{k=1}^{l} \frac{d_k}{s - p_k} \]

where \( d_k \) are residues of poles at \( s = p_k \).

Suppose, transfer function has identical pole and zero at \( z = p \) thus \( d_k = 0 \). The effect of this cancellation on controllability and observability depends on how the state variables are defined.

Example 8.39. Consider the following state equations and check controllability and observability of the given systems and explain why there is a difference in controllability and observability of the same system.

(i) \[ \dot{x}(t) = \begin{bmatrix} 0 & 1 \\ -6 & -5 \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} r(t) \]

\[ c(t) = \begin{bmatrix} 2 & 1 \end{bmatrix} x(t) \]

(ii) \[ \dot{x}(t) = \begin{bmatrix} 0 & -6 \\ 1 & -5 \end{bmatrix} x(t) + \begin{bmatrix} 2 \\ 1 \end{bmatrix} r(t) \]

\[ c(t) = \begin{bmatrix} 0 & 1 \end{bmatrix} x(t) \]

Solution:

(i) \[ A = \begin{bmatrix} 0 & 1 \\ -6 & -5 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad C = \begin{bmatrix} 2 & 1 \end{bmatrix} \]

For controllability \( Q = [B \quad AB] \)

\[ AB = \begin{bmatrix} 0 & 1 \\ 1 & -5 \end{bmatrix}, \quad Q = \begin{bmatrix} 0 & 1 \\ 1 & -5 \end{bmatrix} \]

\( Q = 0, \text{ Rank = 2, system is controllable} \)

\[ C^T = \begin{bmatrix} 2 \\ 1 \end{bmatrix}, \quad A^T = \begin{bmatrix} 0 & -6 \\ 1 & -5 \end{bmatrix} \]

\[ Q' = [C^T \quad A^T C^T] = \begin{bmatrix} 2 & -6 \\ 1 & -3 \end{bmatrix} \]
Rank = 1, system is unobservable.

$A = \begin{bmatrix} 0 & -6 \\ 1 & -5 \end{bmatrix}$, $B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, $C = [0 \ 1]$

For controllability $Q = [B \ AB]$, $AB = \begin{bmatrix} -6 \\ -3 \end{bmatrix}$

$Q = \begin{bmatrix} 2 & -6 \\ 1 & -3 \end{bmatrix}$

Rank = 1, the system is uncontrollable.

For observability $Q' = [C^T \ A^T \ C^T]$

$A^T C^T = \begin{bmatrix} 1 & -8 \\ -9 & 1 \end{bmatrix}$, $Q' = \begin{bmatrix} 0 & 1 \\ 1 & -5 \end{bmatrix}$

Rank = 2, the system is observable.

Now find the transfer function.

$\frac{sI - A}{[sI - A]} = \frac{s}{s + 5} = \frac{1}{s + 5}$

$s + 5 + 5s + 6 = (s + 2)(s + 3)$

$[sI - A]^{-1} = \frac{s + 5}{s^2 + 5s + 6}$

Transfer function is given by

$CT[sI - A]^{-1}B = [0 \ 1]$

$\begin{bmatrix} s + 5 \\ s^2 + 5s + 6 \\ 1 \\ s^2 + 5s + 6 \end{bmatrix}$

Transfer function $= \frac{s + 5}{(s + 2)(s + 3)}$

From the transfer function it is clear that there is a pole at $s = -2$ and a zero at $s = -1$, there is a pole zero cancellation. Therefore for completely controllable and observable the transfer function must not have any pole zero cancellation.

**SUMMARY**

**Advantages of State Space Techniques**

1. This method can be applied to linear or non-linear, time variant or time invariant system.
2. It is easy to apply where laplace transform cannot be applied.
3. It is a time domain approach.
4. This method is suitable for digital computer computation.

**EXERCISE**

1. The state equation of a linear time invariant system is given by

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t)$$

Find the state transition matrix and the characteristic equation.
8.2. Find the state transition matrix for
\[
\begin{align*}
\dot{x}_1 &= x_2 \\
\dot{x}_2 &= -6x_1 - 5x_2
\end{align*}
\]

8.3. The state space of a system is represented by the following equations
\[
\begin{bmatrix}
\dot{x}_1 \\
\dot{x}_2 \\
y
\end{bmatrix} =
\begin{bmatrix}
-3 & 1 \\
-2 & 0 \\
1 & 0
\end{bmatrix}
\begin{bmatrix}
x_1 \\
x_2
\end{bmatrix}
+ \begin{bmatrix}
0 \\
1
\end{bmatrix} u ; \ t > 0
\]
\[
y = \begin{bmatrix}
1 & 0
\end{bmatrix}
\begin{bmatrix}
x_1 \\
x_2
\end{bmatrix}
\]

Find the transfer function of the system.

8.4. Obtain A, B, C and D matrices of the circuit

8.5. The state space representation of a system is
\[
\begin{bmatrix}
\dot{x}_1 \\
\dot{x}_2 \\
\dot{x}_3
\end{bmatrix} =
\begin{bmatrix}
-5 & 2 & 1 \\
0 & 0 & 1 \\
-1 & -4 & -3
\end{bmatrix}
\begin{bmatrix}
x_1 \\
x_2 \\
x_3
\end{bmatrix}
+ \begin{bmatrix}
0 \\
u \\
1
\end{bmatrix} u
\]
\[
y = x_1
\]

Is the system controllable?

8.6. Obtain a state space representation for the system
\[
\frac{d^3y}{dt^3} + 5 \frac{d^2y}{dt^2} + 6 \frac{dy}{dt} + 3y = 6u(t)
\]

8.7. For question no. (8.35) determine the transfer function.

8.8. Obtain the transfer function representation for a system represented by matrices
\[
A = \begin{bmatrix}
0 & 1 & 0 \\
0 & 0 & 1 \\
-10 & -2 & -3
\end{bmatrix} ; \quad B = \begin{bmatrix}
0 \\
0 \\
1
\end{bmatrix} ; \quad C = \begin{bmatrix}
2 & 1 & 0
\end{bmatrix}
\]

8.9. The transfer function of a system is \( G(s) = \frac{2}{(s + 1)(s + 2)} \). Obtain 3 state variable representation

8.10. Obtain the state space representation of the following control system.
8.18. Write the state space representation taking $i_1, i_2, i_3$, and $V_r$ as the variables.

![Fig. 8.32.](image)

8.19. (a) Write the properties of transition matrix.

(b) If \[ A = \begin{bmatrix} 0 & 1 \\ -6 & -5 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad C = \begin{bmatrix} 0 & 1 \end{bmatrix} \]

Determine $\Phi(t)$.

8.20. (a) For the given transfer function, obtain the state model

\[ G(s) = \frac{k}{s^3 + a_2s^2 + a_2s + a_1} \]

OR

consider a control system with the state model given by

\[ \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -2 & -4 & -3 \end{bmatrix} x + \begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix} u \]

\[ X_1(0) \text{ and } X_2(0) = 0 \text{ and } u = \text{unit step function. Compute the state transition matrix and hence find state response.} \]

(b) For the given system check the controllability and the observability of the system

\[ \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -2 & -4 & -3 \end{bmatrix} x + \begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix} u \]

\[ Y = \begin{bmatrix} 0 & 1 & -1 \\ 1 & 2 & 1 \end{bmatrix} x \]

ANSWERS

8.1. \[ \begin{bmatrix} te^{-t} + e^{-t} & te^{-t} \\ -te^{-t} & -te^{-t} + e^{-t} \end{bmatrix}, \text{ characteristic equation } = s^2 + 2s + 1 = 0 \]

8.2. \[ \begin{bmatrix} 3e^{2t} + 2e^{3t} & e^{2t} - e^{3t} \\ -6e^{2t} + 6e^{3t} & -2e^{2t} + 3e^{3t} \end{bmatrix} \]

8.3. \[ \frac{1}{(s+1)(s+2)} \]

8.5. Not controllable

8.6. \[ \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -640 & -172 & -8 \end{bmatrix} x + \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 610 & 160 & 0 \end{bmatrix} u \]

8.7. Not controllable

8.8. \[ \begin{bmatrix} X_1 \\ X_2 \\ X_3 \end{bmatrix} = \begin{bmatrix} \frac{1}{L_1} & \frac{R_1}{L_1} \\ \frac{R_2}{L_2} & \frac{1}{L_2} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \end{bmatrix} u \]

8.9. \[ \begin{bmatrix} V_1 \\ V_2 \end{bmatrix} = \begin{bmatrix} 0 & -0.5 & -0.5 \\ 0.5 & -0.5 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ -0.5 \end{bmatrix} u \]

SEMI-OBJECTIVE TYPE QUESTIONS

(i) What are the advantages of state space techniques.

(ii) Define state, state variables, state vector and state space.

(iii) What is state transition matrix?

(iv) Define Eigen values and Eigen vectors.

(v) What are the properties of STM.

(vi) Define controllability and observability.

(vii) Write a short note on Kalman's Test.

(viii) Write a short note limitations of state variable approach.
Chapter 9

Control System Components

9.1. INTRODUCTION
In control system, the devices which are used to convert the process variables in one form to another form is known as transducers. Transducers can also be defined as a device which transfers the energy from one form to another, for example a thermocouple converts the heat energy into electrical voltage. In control system the following devices are used as a transducer.

1. Potentiometers
2. D.C. servomotors
3. A.C. servomotors
4. Synchrons
5. Stepper motors
6. Magnetic amplifier
7. Tachogenerators
8. Gyroscope
9. Differential transformer

9.2. POTENTIOMETERS
A potentiometer is a simple device which is used for mechanical displacement either linear or angular. Thus, a potentiometer is an electromechanical transducer which converts the mechanical energy into electrical energy. The input to the device is in the form of linear mechanical displacement or rotational mechanical displacement, when the voltage is applied across the fixed terminals the output voltage is proportional to the displacement.

Consider the fig. 9.1 (translational potentiometer).

Let
\( E_1 = \text{input voltage} \)
\( E_0 = \text{output voltage} \)
\( x_1 = \text{displacement from zero position} \)
\( x_T = \text{total length of translational potentiometer} \)
\( R = \text{Total resistance of potentiometer} \)

Under ideal condition the output voltage \( E_0 \) is given by

\[ E_0 = \frac{x_1}{x_T} E_1 \]

9.2. SERVOMOTORS
Servomotors are used in feedback control systems. Servomotors have low rotor inertia and high speed of response. The servomotors are also known as control motors. The servomotors which are used in feedback control system should have linear relationship between electrical control signal and rotor speed, torque speed characteristic should be linear, the response of the servomotor should be fast and inertia should be low.

9.4. TYPES OF SERVOMOTORS
The servomotors are classified as
(i) A.C. servomotors
(ii) D.C. servomotors
(iii) Special servomotors

D.C. servomotors are further classified as armature controlled d.c. servomotors and field controlled d.c. servomotors.
9.4.1. A.C. Servomotors

These motors have two parts, namely stator and rotor. A.C. servomotors are two phase induction motors. The stator has two distributed windings. These windings are displaced from each other by 90° electrical. One winding is called the main winding or reference winding. The other winding is excited by constant a.c. voltage. The other winding is called the control winding. The reference winding is excited by variable control voltage of the same frequency as the reference winding but having a phase displacement of 90° electrical. The variable control voltage for control winding is obtained from a servoamplifier. The direction of rotation depends upon phase relationship of voltages applied to the two windings. The direction of rotation of the rotor can be reversed by reversing the phase difference between control voltage and reference voltage.

The rotor of a.c. servomotors are of two types (a) squirrel cage rotor (b) drag cup type rotor. The squirrel cage rotor having large length and small diameter, so its resistance is very high. The air gap of squirrel cage is kept small. In drag cup type there are two air gaps. For the rotor a ring of non-magnetic conducting material is used. A stationary iron core is placed between the conducting cup to complete the magnetic circuit. The resistance of drag cup type is high and therefore having high starting torque. Generally aluminium is used for cup. Fig. 9.4 shows the schematic diagram of two phase a.c. servomotor and 9.5(a), 9.5(b) shows the two types of rotor.

9.4.2. Torque-speed Characteristic

The torque speed characteristic of two phase induction motor depends upon the ratio of reactance to resistance. For high resistance and low reactance, the characteristic is linear and for large ratio of X to R it becomes non-linear as shown in Fig. 9.6(a). The torque-speed characteristics for various control voltages are almost linear as shown in Fig. 9.6(b).

H.A. D.C. Servomotors

D.C. servomotors are separately excited or permanent magnet d.c. servomotors. The armature of d.c. servomotor has a large resistance, therefore torque speed characteristic is linear. The torque speed characteristic shows in Fig. 9.7(b). Fig. 9.7(a) shows the schematic diagram of separately excited d.c. servomotor.

The d.c. servomotors can be controlled from armature side or from field. In field controlled d.c. servomotors the ratio of L/R is large i.e., the time constant for field circuit is large. Due to large time constant, the response is slow and therefore they are not commonly used. Transfer function of field controlled d.c. servomotor is given in chapter 1. The speed of the motor can be controlled by adjusting the voltage applied to the armature. In armature controlled d.c. servomotor the time constant is small and hence the response is fast. The efficiency is better than the field controlled motor. The transfer function of armature controlled d.c. servomotor is derived in chapter 1.

9.4.4. Application of Servomotors

Servomotors are widely used in radars, electromechanical actuators, computers, machine tools, tracking and guidance system, process controllers and robots.
9.5. TACHOGENERATORS (TACHOMETERS)

Tachometers are electromechanical devices, which transforms the mechanical energy into electrical energy. In tachometers its magnetic flux is constant and induced e.m.f is proportional to the speed of the shaft i.e., angular speed. The tachometers are classified as d.c. tachometers and a.c. tachometers.

9.5.1. D.C. Tachometer

In control systems most common type of tachometers are d.c. tachometers. D.C. tachometer contains an iron core rotor and permanent magnet. The magnetic field is provided by the permanent magnet and no external supply voltage is necessary. The input to the tachometer is the speed of the shaft and the output is voltage which is proportional to the angular speed of the shaft.

\[ e = K \omega(t) \]

where, \( e \) = tachometer generator voltage
\( K \) = tachometer sensitivity
\( \omega \) = angular speed of the shaft.

Laplace transform of equation 9.4

\[ E(S) = K \omega(S) \]

\( E(S) \) = Transfer function of tachometer is

\[ K = \frac{E(S)}{\omega(S)} \]

In d.c. tachometers the windings on rotor are connected to the commutator and the output voltage is taken across the brushes. The permanent magnet tachometers are compact and reliable but having high inertia. For reducing the voltage drop across the brushes, metal brushes with silver tips are used.

9.5.2. Advantages of D.C. Tachometers

1. At zero speed there is no residual voltage.
2. It is possible to generate a very high voltage gradients in small size.
3. It can be used with high pass output filters to reduce servo velocity lags.

9.5.3. A.C. Tachometers

A.C. tachometers are similar to two phase induction motor. The schematic diagram of a.c. tachometer is shown in fig. 9.8. Here two stator windings are placed in quadrature with each other and rotor is short circuited. The rotor of tachometer is a thin aluminium cup. The two stator windings are known as primary winding and secondary winding. Primary winding is known as reference winding.

The primary winding carries a voltage \( V/90^\circ \). This voltage is fixed in magnitude at fixed frequency known as carrier frequency. The coil axis of primary winding is called direct axis. When the current in primary winding flows a pulsating field \( B_1 \) is produced along direct axis. When the rotor is stationary an e.m.f. is induced in it. Since, the motor is short circuited eddy current flow in it. Due to this eddy current a pulsating flux \( B_2 \) is produced opposite to the \( B_1 \). Hence a pulsating flux \( B_1 \) is produced due to the vector sum of \( M_1 \) and \( M_2 \). Since the secondary winding is in quadrature no e.m.f will be produced in it due to \( B_1 \). Now if the rotor rotates it will induce the flux and an e.m.f is induced in it, due to this induced e.m.f a current will flow in the rotor and a pulsating flux will appear along the coil axis of secondary winding. This flux \( \phi_2 \) will induce a voltage in secondary winding. The magnitude of the output voltage will be proportional to the rotor speed.

9.6. STEPPER MOTORS

Stepper motors, the movement of the rotor is in discrete steps. A stepper motor is electromechanical device. There are three types of stepper motors:

(i) Variable reluctance motors
(ii) Permanent magnet motors
(iii) Hybrid type

9.6.1. Variable Reluctance Stepper Motors

Variable reluctance stepper motors are of two types namely single-stack variable reluctance motor and multi-stack variable reluctance motor.


4-phase, 4-stator pole, 2-tooth rotor variable reluctance stepper motor is shown schematically in fig. 9.9.

In fig. 9.9 four phases are connected to d.c. source through switches \( S_1, S_2, S_3, \) and \( S_4 \). When phase 1 is excited, the rotor aligns with the axis of phase 1 as shown in fig. 9.9. If now switch \( S_1 \) is off and \( S_2 \) is switched on, the rotor moves through 90° in clockwise to align with the resultant air gap field which lies along the axis of phase 2. Now, phase 3 is excited and phase 2 is disconnected, the rotor aligns with the resultant air gap field which now lies along phase 3 and so on. The windings of the stator are energized in sequence 1, 2, 3, 4, 1.

![Fig. 9.8. A.C. tachometer](image)

![Fig. 9.9. Variable reluctance stepper motor](image)
9.6.1.2. Multistack Variable Reluctance Stepper Motor

A multistack variable reluctance stepper motor is shown in Fig. 9.10. In this type of stepper motor, rotors have a common shaft and stator have a common frame. The stators and rotors have the same number of poles with same tooth size. The stators are pulse excited while rotors are unexcited. The windings of all stator poles are excited simultaneously. Since stators and rotors have some number of stacks or phases, then tooth pitch is given by $360^{\circ}/T_s$, and angular displacement per step angle is given by $360^{\circ}/nT_s$. For example 12 pole rotor, the pitch is $360^{\circ}/12 = 30^\circ$ and the step angle will be $360/3 \times 12 = 10^\circ$ i.e., rotor poles are displaced from each other by $10^\circ$.

![Fig. 9.10. Multistack variable reluctance stepper motor](image)

9.7. SYNCHROS

A synchro is an electromagnetic transducer which converts the angular position of a shaft into an electric signal. Synchros are used as detectors and encoders.

9.7.1. Synchro Transmitter

The construction of synchro transmitter is very similar to that of a three phase alternator. The stator is made of laminated silicon steel and carries three phase star connected windings. The rotor is a rotating part, dumbbell-shaped magnet with a single winding.

A single phase a.c. voltage is applied to the rotor through slip rings. Let applied a.c. voltage to the rotor be $e_s = E_s \sin \omega t$.

due to this applied voltage a magnetizing current will flow in the rotor coil. This magnetizing current produces sinusoidally varying flux and distributed in the air gap. Because of transformer action voltages get induced in all stator coil which is proportional to cosine of angle between stator and rotor coil axes.

Now, consider the rotor of synchro transmitter is at an angle $\theta$, then voltages in each stator coil with respect to neutral are

\[ E_{a0} = KE_s \sin \omega t \cos \theta \]
\[ E_{b0} = KE_s \sin \omega t \cos (\theta + 120^\circ) \]
\[ E_{c0} = KE_s \sin \omega t \cos (\theta + 240^\circ) \]

Similarly,

\[ E_{a0} = \sqrt{3} KE_s \sin \omega t \sin \theta \]
\[ E_{b0} = \sqrt{3} KE_s \sin \omega t \sin (\theta + 120^\circ) \]
\[ E_{c0} = \sqrt{3} KE_s \sin \omega t \sin (\theta + 240^\circ) \]

When $\theta = 0$, the maximum induced voltage will be $E_{a0}$ and $E_{b0}$ will be zero. This position of the rotor is defined as electrical zero of the transmitter and is used as the reference for indicating the angular position of the rotor.

Thus, the input to the synchro transmitter is the angular position of the rotor shaft and the output are the three single phase voltages which are the function of the shaft position.

9.7.2. Synchro Control Transformer

Principle of operation of synchro control transformer is same as that of synchro transmitter. Each of synchro control transformer is cylindrical type. Synchro control transformer is an electromechanical device. The combination of synchro transmitter and synchro control transformer is used as an error detector. The function of error detector is to convert the difference of two shaft positions into an electrical signal. The Fig. 9.12, shows schematic diagram of synchro error detector.

The output of synchro transmitter is connected to the stator winding of the synchro control transformer. Therefore the same current will flow in the stator windings of synchro control transformer in opposite direction. The voltage across the rotor terminals of control transformer is

\[ e(t) = K_e V_s \cos \phi \sin \omega t \]

where $\phi$ = angular displacement between the two rotors. When the two rotors are at an angle $0^\circ$, the voltage induced in control transformer is zero. This position is known as electrical zero position of control transformer.

![Fig. 9.12. Synchro error detector](image)
\[ \phi = (90^\circ - \theta + \alpha) \]

Put the value of \( \phi \) in equation 9.14, we get

\[ e(t) = K_1 V_r (\theta - \alpha) \sin \omega_0 t \]

From eqn. (9.16) it is clear that when two rotor shafts are not in alignment, the race velocities of control transformer is approximately a sine function of the difference between the two shaft angles.

For small angular displacement between two rotor position

\[ e(t) = K_1 V_r (\theta - \alpha) \sin \omega_0 t \]

**SUMMARY**

For feedback control system, servomotor should have following properties

(i) It should be reversible.
(ii) Response should be fast.
(iii) Relationship between electrical control system and rotor speed should be linear.
(iv) It should have linear torque-speed characteristic.

A.C. Servomotor has following features.

(i) Light in weight.
(ii) Reliable and stable.
(iii) Maintenance free.
(iv) Large \( R \) to \( X \) ratio.
(v) Smooth \& noise free.

Field controlled D.C. motor has following features.

(i) It has large time constant.
(ii) It is open loop system.
(iii) Control circuit is simple.

Table 9.1: Comparison between A.C. and D.C. servomotors

<table>
<thead>
<tr>
<th>S. No.</th>
<th>A.C. Servomotor</th>
<th>D.C. Servomotor</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>Low power output</td>
<td>High power output</td>
</tr>
<tr>
<td>2</td>
<td>Efficiency is less</td>
<td>High efficiency</td>
</tr>
<tr>
<td>3</td>
<td>No brushes &amp; slip rings free</td>
<td>Frequent maintenance required</td>
</tr>
<tr>
<td>4</td>
<td>No radio frequency noise</td>
<td>Brushes produce radio frequency noise</td>
</tr>
<tr>
<td>5</td>
<td>Smooth operation</td>
<td>Noisy operation</td>
</tr>
</tbody>
</table>

**Definitions Related to Stepper motor**

(i) **HOLDING TORQUE**: It is defined as the maximum static torque that can be applied to the shaft of an excited motor without causing a continuous rotation.
(ii) **DETENT TORQUE**: It is defined as the maximum static torque that can be applied to the shaft of an unexcited motor without causing a continuous rotation.
(iii) **STEP ANGLE**: It is defined as the angular displacement of the rotor in response to each pulse.
(iv) **CRITICAL TORQUE**: It is defined as the maximum load torque at which rotor does not move when an exciting winding is energized. This is also known as pull-out torque.

**EXERCISE**

1. Write short note on stepper motor.
2. Explain the working of Tachometers.
3. State the applications of a.c. servomotor.
5. Write short note on potentiometer.
7. Sketch the torque-speed characteristics of A.C. servomotor.

**SEMI-OBJECTIVE TYPE QUESTIONS**

(i) Write short note on potentiometers.
(ii) Distinguish between a.c. servomotor and D.C. servomotor.
(iii) State applications of servomotor.
(iv) What are the advantages of D.C. tachometers.
(v) Write a short note on multistack variable reluctance stepper motor.
(vi) What are the major applications of stepper motor.
(vii) Write short note on synchro transmitter.
(viii) Draw the torque-slip characteristic of D.C. servomotor.
(ix) Draw the torque-slip characteristic of A.C. servomotor.
(x) Write short note on synchro control transformer.
(xi) What is a synchro transmitter?
(xii) Draw the torque-speed characteristics of a two phase servomotor.
(xiii) What are the two varieties of stepper motors?
(xiv) Explain the use of synchros as transmission of position over long distances.
(xv) Distinguish between d.c. motor and d.c. servomotors.
(xvi) What is the function of tachogenerator?
(xvii) Give the applications of A.C. tachogenerator.
(xviii) Explain the use of synchro pair as an error detector.
(xix) What are the applications of synchro?
(xx) Give a trade name of synchro.

Chapter 10
The Laplace Transforms

10.1. INTRODUCTION
In Laplace transform method, the differential equation in the time domain is transformed into s-plane. The complex frequency variable is given by $S = \sigma + j\omega$, where $\sigma$ and $\omega$ are both real quantities, $\sigma$ is the real part and $\omega$ is the imaginary part. The advantage of Laplace transform is that this method gives total solution. For example when we solve the differential equation by Laplace transform, both transient and steady state component of the solution can be obtained simultaneously.

10.2. DEFINITION OF LAPLACE TRANSFORM

One-sided Laplace transform of a function $f(t)$ is given by

$$
\mathcal{L}\{f(t)\} = \int_0^\infty f(t) e^{-st} \, dt
$$

where $f(t)$ is time function defined in the interval $0 \leq t \leq \infty$

$s$ = complex variable

$F(s)$ = Laplace transform of $f(t)$

10.3. LAPLACE TRANSFORMS OF SOME TIME FUNCTIONS

10.3.1. Exponential Function
Consider the exponential function

$$
f(t) = \begin{cases} 
0 & \text{for } t < 0 \\
K e^{-at} & \text{for } t \geq 0 
\end{cases}
$$

where ‘$K$’ and ‘$a$’ are constants.

$$
\mathcal{L}\{K e^{-at}\} = \int_0^\infty K e^{-at} e^{-st} \, dt = K \int_0^\infty e^{-(a+s)t} \, dt
$$

$$
= K \left[ \frac{1}{a+s} e^{-(a+s)t} \right]_0^\infty = \frac{K}{s+a} [e^{-\infty} - e^{0}]
$$

$$
\mathcal{L}\{K e^{-at}\} = \frac{K}{s+a}
$$

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10.3.2. Step Function
Consider the step function
\[ f(t) = \begin{cases} 0 & \text{for } t < 0 \\ A & \text{for } t > 0 \end{cases} \]
where 'A' is a constant.

\[ \mathcal{L}\{f(t)\} = \mathcal{L}\{A\} = \frac{A}{s} \]

10.3.3. Unit Step Function
Consider the unit step function
\[ f(t) = \begin{cases} 0 & \text{for } t < 0 \\ 1 & \text{for } t > 0 \end{cases} \]

\[ \mathcal{L}\{f(t)\} = \mathcal{L}\{1\} = \frac{1}{s} \]

10.3.4. Ramp Function
Consider the ramp function
\[ f(t) = \begin{cases} 0 & \text{for } t < 0 \\ At & \text{for } t \geq 0 \end{cases} \]

\[ \mathcal{L}\{At\} = \int_0^\infty At e^{-st} dt = \frac{A}{s^2} \]

10.3.5. Sinusoidal Function
Consider the sinusoidal function
\[ f(t) = \begin{cases} 0 & \text{for } t < 0 \\ A \sin \omega t & \text{for } t \geq 0 \end{cases} \]

\[ \mathcal{L}\{A \sin \omega t\} = A \mathcal{L}\{\sin \omega t\} = \frac{A \omega}{s^2 + \omega^2} \]

For
\[ A = 1 \]
\[ \mathcal{L}\{\sin \omega t\} = \frac{\omega}{s^2 + \omega^2} \]

10.3.6. Laplace Transform of \( t^n \)
\[ \mathcal{L}\{t^n\} = \int_0^\infty t^n e^{-st} dt \]

Let
\[ dt = \frac{1}{s} dx \]
\[ t = \left( \frac{x}{s} \right) \]

When
\[ t = 0, x = 0 \]
\[ t = \infty, x = \infty \]

\[ \mathcal{L}\{t^n\} = \frac{n!}{s^{n+1}} \int_0^\infty x^n e^{-x} dx \]

\[ n+1 \Delta = \int_0^n x^n e^{-x} dx \]
\[ n+1 = n + \Delta \]

\[ \mathcal{L}\{t^n\} = \frac{1}{s^{n+1}} n! = \frac{n!}{s^{n+1}} \]

10.3.7. Hyperbolic Sine and Cosine Functions
\[ f(t) = \cosh \beta t \]
\[ \mathcal{L}\{f(t)\} = \mathcal{L}\{\cosh \beta t\} = \frac{1}{s^2 - \beta^2} \]
\[ \cosh \beta t = \frac{e^{\beta t} + e^{-\beta t}}{2} \]
\[ \mathcal{L}\{e^{\beta t}\} = \frac{1}{s - \beta} \]
\[ \mathcal{L}\{e^{-\beta t}\} = \frac{1}{s + \beta} \]
\[ \mathcal{L}\{\cosh \beta t\} = \frac{s}{s^2 - \beta^2} \]
\[ \mathcal{L}\{\sinh \beta t\} = \frac{s}{s^2 + \beta^2} \]

Similarly,
\[ \mathcal{L}\{\sinh \alpha t\} = \frac{\alpha}{s^2 + \alpha^2} \]
\[ \mathcal{L}\{\cosh \alpha t\} = \frac{s}{s^2 - \alpha^2} \]
10.3.8. Damped Sine and Cosine Functions

\[ f(t) = e^{-at} \sin \beta t \]

\[ F(s) = \frac{\alpha}{(s+a)^2 + \beta^2} \]

Example 10.1. Find the Laplace transform of

\[ f(t) = e^{at} \sin \omega t \]

Solution:

\[ \mathcal{L} \{ e^{at} \sin \omega t \} = \frac{\omega}{(s-a)^2 + \omega^2} \]

\[ \mathcal{L} \{ e^{at} \cos \beta t \} = \frac{s+a}{(s+a)^2 + \beta^2} \]

PROPERTIES OF LAPLACE TRANSFORMS

1. If \( \phi \) is a constant and \( F(s) \) is the Laplace transform of \( f(t) \) then

\[ \mathcal{L} \{ \phi f(t) \} = \phi \mathcal{L} \{ f(t) \} \]

2. If \( f_1(t) \) and \( f_2(t) \) are the time functions and \( F_1(s) \), \( F_2(s) \) are their Laplace transforms then

\[ \mathcal{L} \{ f_1(t) \pm f_2(t) \} = \mathcal{L} f_1(t) \pm \mathcal{L} f_2(t) = F_1(s) \pm F_2(s) \]

3. Time shift property: If \( \alpha \) is a positive real number and \( F(s) \) is the Laplace transform of \( f(t) \) then

\[ \mathcal{L} \{ f(t-a) \} = e^{-\alpha a} F(s) \]

4. Differentiation theorem:

The Laplace transform of the derivative of a function \( f(t) \) is given by

\[ \mathcal{L} \left\{ \frac{d^n f(t)}{dt^n} \right\} = s^n F(s) - f^{(n-1)}(0) \]

where \( f(0) \) is the initial value of \( f(t) \) at \( t = 0 \)

5. Laplace transform of integral of \( f(t) \)

\[ \mathcal{L} \{ \int f(t) \ dt \} = \frac{F(s)}{s} \]

where \( F(s) \) is the Laplace transform of \( f(t) \) and

\[ \mathcal{L} \{ f(t) \} = \frac{F(s)}{s} \]

where \( F(s) \) is the Laplace transform of \( f(t) \) and
8. Complex differentiation theorem:

\[ L \left[ a f(t) \right] = (a-s)^{n} F(s) \]

\[ e.g., \text{Find the Laplace transform of } t \sinh at \]

\[ L \left[ t \sinh at \right] = \frac{2as}{(s^{2} - a^{2})^{2}} \]

9. Complex integration:

If \( F(s) \) is the Laplace transform of \( f(t) \) then

\[ L \left[ \frac{f(t)}{t} \right] = \int_{0}^{\infty} F(s) \frac{ds}{s} \]

\[ e.g., \text{find the Laplace transform of } \sin \frac{2t}{t} \]

\[ L \sin \frac{2t}{t} = \frac{2}{s^{2} + 4} \]

\[ \frac{2}{s^{2} + 4} ds = 2 \cdot \frac{1}{2} \left( \tan^{-1} \frac{s}{2} \right) \]

\[ = \left( \tan^{-1} \infty - \tan^{-1} \frac{s}{2} \right) = \frac{\pi}{2} - \tan^{-1} \frac{s}{2} \]

10. Initial value theorem:

If initial value of the function \( f(t) \) is required, then it can be calculated by initial value theorem which states

\[ \lim_{t \to 0} f(t) = \lim_{s \to \infty} s F(s) \]

\[ e.g., \text{Find the initial value of } e^{-at} \]

\[ f(t) = e^{-at} \]

\[ F(s) = Lf(t) = \frac{1}{s + a} \]

\[ \lim_{t \to 0} f(t) = \lim_{s \to \infty} s \cdot \frac{1}{s + a} = \lim_{s \to \infty} \frac{1}{s + a} = 1 \]

10.5. INVERSE LAPLACE TRANSFORM

If \( F(s) \) is the Laplace transform of \( f(t) \) then

\[ L^{-1} F(s) = f(t) = \frac{1}{2\pi j} \int_{c-j\infty}^{c+j\infty} F(s) e^{st} ds \]

But this method is not generally used. In practice if \( F(s) \) is a rational function obtained the partial fraction and take the inverse Laplace with the help of standard table of transform pairs.

Example 10.2. Find the inverse Laplace transform of \( \frac{2}{s(s+2)} \).

Solution: Given: \( F(s) = \frac{2}{s(s+2)} \)

The partial fraction of \( F(s) \) is

\[ F(s) = \frac{1}{s} - \frac{1}{s+2} \]

\[ f(t) = L^{-1} F(s) = L^{-1} \frac{1}{s} - L^{-1} \frac{1}{s+2} = 1 - e^{-2t} \]

Ans.

10.6. APPLICATION OF LAPLACE TRANSFORM IN CONTROL SYSTEM

In control systems the system is described by the integro-differential equations. It is very difficult to solve the such equations in time domain. The laplace transforms of such equations converts them into simple algebraic equations. Then by using the inverse laplace time domain function can be obtained.

Example 10.3. Find \( i_{2} \) by using Laplace transformation.

Solution: For mesh 1:

\[ \frac{di_{1}}{dt} + 20i_{1} - 10i_{2} = 100 u(t) \]

\[ \frac{di_{2}}{dt} + 10i_{2} = 0 \]
Laplace transform of (10.4) & (10.5)

\[ \frac{sl_1(s) + 20l_1(s) - 10l_2(s)}{s} = \frac{100}{s} \]

\[-10l_1(s) + s l_2(s) + 20l_2(s) = 0 \]
equation (10.6) and (10.7) can be written as

\[ l_1(s) [s + 20] - 10l_2(s) = \frac{100}{s} \]

\[-l_1(s) + 10s + (s + 20) l_2(s) = 0 \]

Solve (10.8) and (10.9)

\[ l_2(s) = \frac{1000}{s(s^2 + 40s + 300)} \]

or,

\[ l_2(s) = \frac{1000}{s^2 + 40s + 300} \]

= \frac{3.33}{s} - \frac{5}{s + 10} + \frac{1.67}{s + 30} \]

Inverse laplace of 10.11

\[ \mathcal{L}^{-1} l_2(s) = l_2(t) = \frac{3.33}{s} - \frac{5}{s + 10} + \frac{1.67}{s + 30} \]

= \frac{3.33 - 5e^{-10t} + 1.67e^{-30t}}{s} \]

Ans.

Example 10.4. Obtain the solution of differential equation given below

\[ \frac{d^2y(t)}{dt^2} + 3 \frac{dy(t)}{dt} + 2y(t) = 5 \]

assuming zero initial conditions.

Solution: \[ s^2 y(s) + 3sy(s) + 2y(s) = \frac{5}{s} \]

\[ y(s) = \frac{5}{s(s^2 + 3s + 2)} \]

\[ y(s) = \frac{5}{s(s+1)(s+2)} = \frac{2.5}{s} - \frac{5}{s+1} + \frac{2.5}{s+2} \]

\[ \mathcal{L}^{-1} y(s) = y(t) = \mathcal{L}^{-1} \left( \frac{2.5}{s} - \frac{5}{s+1} + \frac{2.5}{s+2} \right) \]

\[ y(t) = 2.5u(t) - 5e^{-t} - 2.5e^{-2t} \]

Ans.

Example 10.5. Find the inverse Laplace transform of

\[ \frac{s+4}{s(s-1)(s^2+4)} \]

Solution: Resolve the given function into partial fraction

\[ \frac{s+4}{s(s-1)(s^2+4)} = \frac{A}{s} + \frac{B}{s-1} + \frac{Cs+D}{s^2+4} \]

Example 10.6. What is Laplace transform of the function \( f(t) \) shown in Fig. 10.1.

Solution: The given waveform is a straight line, so, first calculate \( f(t) \) using the equation.

\[ y - y_1 = \frac{y_2 - y_1}{x_2 - x_1} (x - x_1) \quad (10.12) \]

From Fig. 10.1:

\[ x_1 = a, \quad x_2 = a + b \]

\[ y_1 = 0, \quad y_2 = b \]

Put all these values in eq. 10.12, we get

\[ f(t) = \frac{b}{a} \]

Ans.

Example 10.7. Obtain the Laplace transform of the waveform shown in Fig. 10.3.

Solution: The given waveform can be divided into two parts

\[ f(t) = f_1(t) + f_2(t) \]

\[ f_1(t) \] can be calculated with the help of equation 10.12

\[ f_1(t) = \frac{T}{T} \quad 0 < t < T \]

\[ f_2(t) \] is shifted unit step function given by

\[ f_2(t) = T u(t-T) \]

\[ f(t) = \begin{cases} 0 & 0 < t < T \\ T & T < t < \infty \end{cases} \]

Ans.
Laplace transform of 10.14 will be

\[ F_1(s) = \int_0^t e^{st} dt = \frac{1}{s} \left( 1 - e^{-st} \right) \]

\[ F_2(s) = \mathcal{L}\{T u(t - T)\} = e^{-st} \frac{1}{s} \]

\( s = F_1(s) + F_2(s) = \frac{1}{s} \left( 1 - e^{-st} \right) \) Ans.

Example 10.8. Obtain the inverse Laplace transform of the following function.

\[ F(s) = \frac{5e^{-s}}{s + 1} \]

Solution: we know that

\[ \mathcal{L}\{e^{st}\} = \frac{1}{s + a} \]

and

\[ \mathcal{L}\{f(t - a)\} = e^{-as} F(s) \]

\[ F(s) = 5e^{-s} \]

Take the inverse Laplace transform

\[ f(t) = 5 e^{(t - 1)} \] Ans.

Example 10.9. Find the Laplace transform of the given function

\[ F_0(s) = \frac{50}{s(s + 2)(s + 0.5)} \]

Solution: Expand by partial fraction

\[ \frac{50}{s(s + 2)(s + 0.5)} = \frac{A}{s} + \frac{B}{s + 2} + \frac{C}{s + 0.5} \]

\[ A = \frac{50}{(s + 2)(s + 0.5)} \]

Similarly,

\[ B = 16.67 \]

\[ C = 66.67 \]

\[ F_0(s) = \frac{50}{s(s + 2)(s + 0.5)} = \frac{50}{s} + \frac{16.67}{s + 2} + \frac{-66.67}{s + 0.5} \]

The inverse Laplace of above eqn

\[ f_0(t) = 50 + 16.67 e^{-2t} - 66.67 e^{-0.5t} \] Ans.

Example 10.10. Find the inverse Laplace transform of

(a) \[ F_1(s) = \frac{6s + 3}{s} \]

(b) \[ F_2(s) = \frac{5s + 2}{(s + 1)(s + 2)^2} \]

Using time shift property from art. 10.4

Example 10.11. Find the inverse Laplace transform of

\[ F(s) = \frac{1}{s^4(s^2 + \omega^2)} \]

Solution: Expand \( F(s) \) by partial fraction

\[ F(s) = \frac{1}{s^4(s^2 + \omega^2)} = \frac{A}{s} + \frac{B}{s^2} + \frac{Cs + D}{s^2 + \omega^2} \]

Comparing the coefficients of \( s^3, s^2, s, \) we have

\[ A + C = 0, \]

\[ B + D = 0, \]

\[ \omega^2 B = 1 \]

Thus:

\[ A = 0 \]

\[ B = 1/\omega^2 \]

\[ C = 0 \]

\[ D = -1/\omega^2 \]

\[ F(s) = \frac{1}{\omega^2 s^3} + \frac{1}{\omega(s^2 + \omega^2)} \]

Inverse Laplace transform of above equation
Example 10.12. Obtain the inverse Laplace transform of the given function

\[ F(s) = \frac{\omega_n^2}{s(s^2 + 2\zeta \omega_n s + \omega_n^2)} \]

Solution: Expand the given function \( F(s) \) by partial fraction.

\[ \frac{\omega_n^2}{s(\zeta^2 + \omega_n^2)} = \frac{A}{s} + \frac{Bs + C}{s^2 + 2\zeta \omega_n s + \omega_n^2} \]

Comparing the coefficients, we have

\[ A + B = 0 \]
\[ 2\zeta \omega_n A + C = 0 \]
\[ \omega_n^2 A = \omega_n^2 \]

\[ A = 1, \quad B = -1, \quad C = -2\zeta \omega_n \]

\[ F(s) = \frac{1}{s} - \frac{2\zeta \omega_n}{s^2 + 2\zeta \omega_n s + \omega_n^2} = \frac{1}{s} - \frac{s + \zeta \omega_n}{s^2 + 2\zeta \omega_n s + \omega_n^2} \]

\[ \frac{1}{s} - \frac{1}{s^2 + 2\zeta \omega_n s + \omega_n^2} = \frac{1}{s} - \frac{1}{s^2 + 2\zeta \omega_n s + \omega_n^2} \]

Since, \( 1 - \zeta^2 \) is a positive quantity for \( 0 < \zeta < 1 \)

\[ \sqrt{1 - \zeta^2} \] is also a positive quantity.

\[ F(S) = \frac{1}{s} - \frac{s + \zeta \omega_n}{s^2 + 2\zeta \omega_n s + \omega_n^2} \left[ \frac{\zeta \omega_n \sqrt{1 - \zeta^2}}{\sqrt{1 - \zeta^2}} \right] \]

Taking the inverse Laplace transform

\[ f(t) = 1 - e^{-\zeta \omega_n t} \cos \left( \omega_n \sqrt{1 - \zeta^2} t \right) - \frac{\zeta \omega_n}{\sqrt{1 - \zeta^2}} e^{-\zeta \omega_n t} \sin \left( \omega_n \sqrt{1 - \zeta^2} t \right) \]

Example 10.13. Obtain the solution of the differential equation

\[ \ddot{x} + \alpha \dot{x} = A \sin \omega_t \quad x(0) = b \]

Solution: The given differential equation

\[ \ddot{x} + \alpha \dot{x} = A \sin \omega_t \]

The forcing function \( e^{-t} \) is given at \( t = 0 \) when the system is at rest.

Solution: The given equation is

\[ \frac{d^2 x(t)}{dt^2} + \alpha \frac{dx(t)}{dt} + 10 x(t) = e^{-t} \]

Taking Laplace transform on both sides of equation (10.16)

\[ s^2 X(s) - sx(0) - x(0) = \frac{1}{s+1} \]

\[ \text{Example 10.14. Solve the differential equation} \]

\[ \frac{d^2 x(t)}{dt^2} + 2 \frac{dx(t)}{dt} + 10 x(t) = e^{-t} \]

when

\[ x(0) = 0 \]
\[ \frac{dx(t)}{dt} = 0 \]

The forcing function \( e^{-t} \) is given at \( t = 0 \) when the system is at rest.
Put the initial conditions
\[ s^2X(s) + 2sX(s) + 10X(s) = \frac{1}{s+1} \]
\[ \Rightarrow \]
\[ X(s) = \frac{1}{(s+1)(s^2+2s+10)} \]

Expand the equation 10.17 by partial fraction
\[ \frac{1}{(s+1)(s^2+2s+10)} = \frac{A}{s+1} + \frac{Bs+C}{s^2+2s+10} \]
\[ 1 = A(s^2 + 2s + 10) + (Bs + C)(s + 1) \]
Comparing the coefficients of \( s^2, s \) and \( s^0 \) we have
\[ A + B = 0 \]
\[ 2A + B + C = 0 \]
\[ 10A + C = 1 \]
\[ A = 1/9, B = -1/9 \text{ and } C = -1/9 \]

Putting all these values in equation 10.18 we get
\[ X(s) = \frac{1}{9(s+1)} - \frac{s+1}{9[(s+1)^2 + 9]} \]

Using,
\[ \mathcal{L}[e^{-at}] = \frac{1}{s+a} \]
\[ \mathcal{L}[e^{-at}f(t)] = F(s+a) \]
\[ \mathcal{L}[e^{as} \cos bt] = \frac{s-a}{(s+a)^2 + b^2} \]

Taking the inverse Laplace of equation (10.19)
\[ x(t) = \frac{1}{9} e^{-t} - \frac{1}{9} e^{-t} \cos 3t \quad \text{Ans.} \]

Example 10.15. For the circuit shown in fig. 10.3, find the current at \( t > 0 \). The switch is closed at \( t = 0 \). Assume no initial charge on the capacitor.

Solution: From the given fig. \( Ri + \frac{1}{C} \int_0^t i \, dt = 50 \)

\[ 10i + \frac{1}{2 \times 10^{-4}} \int_0^t i \, dt = 50 \quad (10.20) \]

Laplace transform of eqn. (10.20)
\[ 10I(s) + \frac{1}{2 \times 10^{-4}} I(s) = \frac{50}{s} \]
\[ I(s) \left[ \frac{s(2 \times 10^{-3}) + 1}{s(2 \times 10^{-4})} \right] = \frac{50}{s} \]
\[ \text{(because for } t < 0 \text{ there is no charge on capacitor)} \]

Example 10.16. In RL circuit the input is given by \( V_S = 10t \). Determine the response of the circuit i.e., \( i(t) \) when \( R = 50 \, \Omega \) and \( L = 5H \).

Solution: The differential equation of the given circuit
\[ Ri + L \frac{di}{dt} = V_s \]
\[ 50i + 5 \frac{di}{dt} = 10t \quad (10.22) \]

Laplace transform of eqn. 10.22
\[ 50I(s) + 5 \frac{dI}{dt} = \frac{10}{s^2} \]
\[ I(s) = \frac{2}{s^2(s+10)} \quad (10.23) \]

Expand the equation 10.23 by partial fraction
\[ I(s) = \frac{2}{s^2(s+10)} = \frac{A}{s} + \frac{B}{s^2} + \frac{C}{s+10} \]
\[ 2 = A \quad s(s+10) + B(s+10) + Cs^2 \]
\[ 10A + B = 0 \]
\[ A + C = 0 \]
\[ 10B = 2 \]
\[ B = 0.2 \]
\[ A = -0.02 \]
\[ C = 0.02 \]

Equation 10.24 becomes
\[ I(s) = \frac{-0.02}{s} + \frac{0.2}{s^2} + \frac{0.02}{s+10} \quad (10.25) \]

Inverse Laplace transform of 10.15
\[ i(t) = -0.02e^{-0.2t} + 0.02e^{-10(t-t_0)} \quad \text{Ans.} \]
\[ i(t) = 0.02e^{-10(t-t_0)} + 0.2(t-t_0) - 0.02e^{-0.2(t-t_0)} \]

Example 10.17. Express the pulse shown in fig. 10.5 as a combination of step functions and obtain its Laplace transform.

Solution:
\[ f(t) = f_1(t) + f_2(t) + f_3(t) \]
\[ f_1(t) = Au_1[t-(t_0)] \quad \text{(shifted step function)} \]
\[ f_2(t) = Au_2[t-(t_0+T)] \]
\[ f_3(t) = Au_1[t-(t_0)] - Au_2[t-(t_0+T)] \]
\[ Ef(t) = E[Au_1(t-t_0)] - E[Au_2(t-t_0+T)] \]
10.7. LAPLACE TRANSFORM OF PERIODIC FUNCTIONS

Theorem: The laplace transform of a periodic function with time period \( T \) is equal to the product of \( \frac{1}{1-e^{-sT}} \) and laplace transform of the first cycle.

Example 10.18. Find the laplace transform of the periodic triangular wave in fig. 10.6.

Solution: To find laplace transform of a periodic wave, first evaluate \( f(t) \) for one cycle (i.e., waveform shown in fig. (10.6a)).

\[
0 < t < T/2 \\
\frac{2E}{T} \left( t - \frac{T}{2} \right) \quad T/2 < t < T
\]

\[
\begin{align*}
\frac{d}{dt} f(t) &= 0 \\
&= \frac{2E}{T} \left( t - \frac{T}{2} \right)
\end{align*}
\]

\[
\begin{align*}
f_1(t) &= \frac{E - 0}{T/2 - 0} (t - 0) = \frac{2E}{T} t \\
&= \frac{2E}{T} \left( t - \frac{T}{2} \right)
\end{align*}
\]

The waveform of equation 10.26 is shown in fig. 10.6(b).

\[
\begin{align*}
\frac{d}{dt} g(t) &= \frac{2E}{T} \delta(t) - \frac{4E}{T} \delta(t - T/2) + \frac{2E}{T} \delta(t - T) \\
&= \frac{2E}{T} (1 - 2e^{-sT/2} + e^{-sT})
\end{align*}
\]

Therefore, Laplace transform of periodic function will be

\[
G(s) = G_1(s) \left( \frac{1}{1-e^{-sT}} \right)
\]

\[
G(s) = \frac{2E}{sT(1 - e^{-sT})} \left( 1 - 2e^{-sT/2} + e^{-sT} \right)
\]

Since

\[
\frac{e^x + e^{-x}}{2} = \cosh x
\]

\[
e^x + e^{-x} = 2 \cosh x
\]

then the eqn 10.29 can be written as

\[
G(s) = \frac{2E}{sT} \left[ \frac{e^{sT/2} + e^{-sT/2} - 2}{e^{sT/2} - e^{-sT/2}} \right]
\]

\[
= \frac{2E}{sT} \left[ \frac{2 \cosh sT/2 - 1}{2 \sinh sT/2} \right]
\]
Also, \( \cosh x - 1 = 2 \sinh^2 \frac{h x}{2} \)

\[
G(s) = \frac{2E}{s^2 T} \frac{2 \sinh^2 s T / 4}{2 \sinh T / 4 \cosh s T / 4} = \frac{2E}{s^2 T} \tanh s T / 4
\]

**Example 10.19.** For the given series RL circuit shown in fig. 10.7 find the current at \( t > 0 \) if the initial current is zero.

**Solution:**

\[
\frac{di}{dt} + 50i = 50 \sin 20 t
\]

or, \( \frac{di}{dt} + 50i = 500 \sin 20 t \)

Take laplace transform of equation (10.31)

\[
sI(s) + 50I(s) = \frac{20}{s^2 + 400}
\]

\[
I(s) = \frac{10000}{(s + 50)(s^2 + 400)} = \frac{A}{s + 50} + \frac{Bs + C}{s^2 + 400} \tag{10.32}
\]

from (10.34) we get

\[
A = 3.4, \quad B = -3.4, \quad C = 172.4
\]

then the equation 10.34 becomes

\[
\frac{10000}{(s + 50)(s^2 + 400)} = \frac{3.4}{s + 50} - \frac{3.4 - s}{s^2 + 400} + \frac{172.4}{s^2 + 400}
\]

\[
= \frac{3.4}{s + 50} - \frac{3.4 - s}{s^2 + 400} + 8.62 \frac{20}{s^2 + 400}
\]

Take inverse laplace transform of above eqn.

\[
l(t) = 3.4 e^{-50t} - 3.4 \cos 20 t + 8.62 \sin 20 t \quad \text{Ans.}
\]

**10.8. CONVOLUTION THEOREM**

If

\[
E \left[ f_1(t) \right] = F_1(s) \text{ and } E \left[ f_2(t) \right] = F_2(s)
\]

then

\[
E \left[ \int_0^t f_1(t - \tau) f_2(\tau) d\tau \right] = \int_0^t F_1(\tau) F_2(t - \tau) d\tau = F_1(s) F_2(s)
\]

**Example 10.20.** Find \( \mathcal{L}^{-1} \left[ \frac{1}{(s^2 + a^2)^2} \right] \)

**Solution:**

\[
F_1(s) = F_2(s) = \frac{1}{s^2 + a^2}
\]

**Example 10.21.** Find \( i_1 \) by using laplace transformation.

**Solution:** For mesh (1)

\[
\frac{di_1}{dt} + 20i_1 - 10i_2 = 100 u(t) \quad \text{...(10.35) 100a(t)}
\]

For mesh (2)

\[
\frac{di_2}{dt} + 20i_2 - 10i_1 = 0 \quad \text{...(10.36)}
\]

The initial values of \( i_1 \) and \( i_2 \) are zero. The laplace transform of eqn 10.35 and 10.36.

\[
(s + 20) I_1(s) - 10I_2(s) = \frac{100}{s}
\]

\[
-10I_1(s) + (s + 20) I_2(s) = 0
\]

Solve 10.37 and 10.38 for \( I_2(s) \)

\[
I_2(s) = \frac{1000}{s(s^2 + 40s + 300)} \tag{10.39}
\]

Expand (10.39) by partial fraction

\[
I_2(s) = \frac{3.33}{s} + \frac{5}{s + 10} + \frac{1.67}{s + 30} \tag{10.40}
\]

Taking inverse laplace transform of eqn 10.40

\[
\mathcal{L}^{-1} I_2(s) = 3.33 e^{-10t} + 1.67 e^{-30t} \quad \text{Ans.}
\]

**Example 10.22.** Solve the differential equation

\[
y(t) + 4y(t) + 3y(t) = 6\quad \text{if} \quad y(0) = y(0) = 0
\]

**Solution:** The given equation is

\[
y(t) + 4y(t) + 3y(t) = 6
\]

Solve the laplace transform of eqn. 10.41

\[
\left[ y(t) - y(0) \right] + 4 \left[ sY(s) - y(0) \right] + 3Y(s) = \frac{6}{s}
\]

\[
\frac{d^2 y(t)}{dt^2} - 3 \frac{dy(t)}{dt} + 2y(t) = 6 \quad \text{Ans.}
\]
Put the initial values
\[ Y(s) (s^2 + 4s + 3) = \frac{6}{s} \]
\[ Y(s) = \frac{6}{s(s^2 + 4s + 3)} \]
Expand the equation (10.42) by partial fraction
\[ Y(s) = \frac{2}{s + 1} - \frac{s + 1}{s + 3} \]
Take inverse laplace of eqn (10.43)
\[ y(t) = 2 - 3e^{-t} + e^{-3t} \]  \text{Ans.}

**Example 10.23.** A voltage pulse of unit height and width \( T \) is applied to the RC circuit at \( t = 0 \). Determine the voltage across the capacitance \( C \) as a function of time.

Solution: From fig. 10.9 (b)

\[ v_i(t) = u(t) - u(t - T) \]  \text{Now at } t = 0, \text{ switch will be closed, the circuit is shown in fig 10.9 (c)}

Applying KVL in the circuit, we have
\[ v_i(t) = Ri(t) + \frac{1}{C} \int_{-\infty}^{t} i(t) \, dt \]
\[ \frac{1}{C} \int_{-\infty}^{t} i \, dt = \frac{q}{C} \]
\[ i = \frac{dq}{dt} = C \frac{dv_i}{dt} \]
\[ v_i(t) = RC \frac{dv_i}{dt} + v_i(t) \]
\[ u(t) - u(t - T) = RC \frac{dv_i}{dt} + v_i(t) \]  \text{Taking Laplace transform on both sides}
\[ \frac{1}{s} - \frac{e^{-sT}}{s} = RC \left[ v_i(s) - v_i(0) \right] + V_C(s) \]
\[ \frac{1}{s} - \frac{e^{-sT}}{s} = (1 + sCR) V_i(s) \]  \text{eqn (10.49)}

Taking inverse laplace of equation (10.49)
\[ v_i(t) = u(t) - \frac{e^{-t/RC} u(t) - u(t - T) + e^{-(t-T)/RC} u(t - T)}{1 - e^{-(t-T)/RC}} \]
\[ \text{The graph of eqn 10.50 is shown in fig. 10.9(d)} \]

**Example 10.24.** The circuit shown in fig. 10.10 is initially under steady-state condition. The circuit is moved from position 1 to position 2 at \( t = 0 \). Find the current after switching.

Solution: Consider the circuit when switch is at position 1 from fig 10.10(a).

\[ i_L(0) = \frac{V}{R_1} \]  \text{eqn (10.51)}

Now, when the switch at position 2, the equivalent circuit will be

\[ L \frac{di}{dt} + (R_1 + R_2) i = 0 \]  \text{eqn (10.52)}

Taking the inverse laplace of eqn (10.52)
\[ L s^2 - L (O_1 - R_1) (R_1 + R_2) \frac{s}{sL + R_1 + R_2} \]
\[ I(s) = \frac{i(t)}{s} \frac{1}{R_1 + R_2} \]

Taking inverse laplace of eqn 10.53

\[ i(t) = i(t) \frac{1}{s} \frac{e^{\frac{-t}{R_1 C}}}{s} \]

Put the value of \( i(t) \) from eqn (10.51) Hence required value of \( i(t) \) will be

\[ i(t) = \frac{V}{R_1} e^{\frac{t}{R_1 C}} \]

Ans.

Example 10.25. Find the Laplace transform of the waveform shown in fig. 10.11.

\[ f(t) \]

![Waveform Image]

Fig. 10.11.

Solution: For one cycle time period = 2

Apply theorem given in art. 10.7

\[ E \int f(t) \frac{1}{1-e^{-st}} \]

\[ F(s) = \frac{1}{1-e^{-2st}} \int_0^{2t} e^{-st} dt + \int_0^{2t} (-1) e^{-st} dt \]

\[ = \frac{1}{1-e^{-2st}} \left( 1 - e^{-2st} \right) \]

\[ F(s) = \frac{s}{1-e^{-2st}} \left( 1 - e^{-2st} \right) \]

\[ F(s) = \frac{1}{s} \left( 1 - e^{-2st} \right) \]

\[ \frac{s}{1-e^{-2st}} \left( 1 - e^{-2st} \right) \]

\[ F(s) = \frac{1}{s} \left[ 1 - e^{-2st} \right] \]

\[ F(s) = \frac{1}{s} \left[ 1 - e^{-2st} \right] \]

\[ \frac{1}{s} \left[ 1 - e^{-2st} \right] \]

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\[ \frac{1}{s} \left[ 1 - e^{-2st} \right] \]

10.8 GATE FUNCTIONS

A rectangular pulse starting at \( t = t_1 \) and duration of pulse is \( T \) having unit height is represented as

\[ C(t) = u(t - t_1) - u(t - t_1 - T) \]

is known as gate function.

Any function multiplied by a gate function will have zero value outside the duration of the gate \( t_1 < t < t_1 + T \) and the value of the function will be unaffected within the duration of the gate \( t_1 < t < t_1 + T \). The gate function is also known as rectangular function.

Example 10.26. Determine the Laplace transform of saw-tooth waveform shown in fig. 10.13(a).

\[ f(t) \]

![Waveform Image]

Fig. 10.13(a)

Solution:

\[ f(t) = \frac{E}{T} t \quad 0 < t < T \]

We can evaluate laplace transform of \( f(t) \) by replacing the time limits by gate function as shown in fig. 10.13(b)

\[ G(t) = u(t) - u(t - T) \]

Hence, the gate function starts from \( t = 0 \) to \( t = T \) and \( f(t) \) can be represented as

\[ f(t) = \frac{E}{T} t \times G(t) \]

\[ f(t) = \frac{E}{T} t \left[ u(t) - u(t - T) \right] = \frac{E}{T} t u(t) - \frac{E}{T} t u(t - T) \]

\[ = \frac{E}{T} t u(t) - \frac{E}{T} (t - T) u(t - T) \]

\[ = \frac{E}{T} t u(t) - \frac{E}{T} (t - T) u(t - T) \]

In the second term we cannot apply time shifting property as the whole function \( u(t - T) \) is not of form \( u(t - a) \)
\( f(t) = \frac{E}{T} u(t) e^{-(t - T)} \) ... 

Taking Laplace transform of eqn (10.59) we get

\[ F(s) = \frac{E}{T} \left( \frac{1}{s} - \frac{e^{-st}}{s} \right) - \frac{E}{T} \] ... (10.60)

** Example 10.27. Find the Laplace transform of following half cycle sine wave. **

** Solution: ** \( f(t) \) is half cycle of sine wave (whose period was \( T \)) can be express as

\[ f(t) = A \sin \left( \frac{2\pi}{T} t \right) \quad 0 < t < T/2 \] ... (10.61)

The time limits can be replaced by the use of gate function.

\[ f(t) = A \sin \left( \frac{2\pi}{T} t \right) \times \text{Ga}(t) \]

\( \text{Ga}(t) \) is from 0 to \( T/2 \) i.e.,

\[ \text{Ga}(t) = u(t) - u(t - T/2) \]

\[ f(t) = A \sin \left( \frac{2\pi}{T} t \right) [u(t) - u(t - T/2)] \]

\[ = A \sin \left( \frac{2\pi}{T} t \right) u(t) - A \sin \left( \frac{2\pi}{T} t \right) u(t - T/2) \]

Now,

\[ = A \sin \left( \frac{2\pi}{T} t \right) u(t - T/2) \]

\[ = A \sin \left( \frac{2\pi}{T} \left( t - T/2 \right) \right) u(t - T/2) \]

\[ = A \left[ \sin \left( \frac{2\pi}{T} \left( t - T/2 \right) \right) \cos \left( \frac{2\pi}{T} \left( t - T/2 \right) \right) + \cos \left( \frac{2\pi}{T} \left( t - T/2 \right) \right) \sin \left( \frac{2\pi}{T} \left( t - T/2 \right) \right) \right] u(t - T/2) \]

\[ = A \sin \left( \frac{2\pi}{T} \left( t - T/2 \right) \right) [-1] u(t - T/2) = -A \sin \left( \frac{2\pi}{T} \left( t - T/2 \right) \right) u(t - T/2) \]

\[ f(t) = A \sin \left( \frac{2\pi}{T} t \right) u(t) + A \sin \left( \frac{2\pi}{T} \left( t - T/2 \right) \right) u(t - T/2) \] ... (10.62)

** Example 10.28. Resolve \( f(t) \) in terms of step, impulse and ramp functions. Also find the laplace transform of \( f(t) \). **

** Solution: ** From fig. 10.15(a)

\[ f(t) = \begin{cases} 
1 & 0 \leq t < 1 \\
4 - t & 1 \leq t < 3 \\
6 & t = 3 \\
2 & 3 \leq t < 4 \\
5 - t & 4 \leq t \leq 5 
\end{cases} \] ... (10.63)

\[ f(t) = \begin{cases} 
[u(t) - u(t - 1)] + (4 - t) [u(t - 1) - u(t - 3)] + 4.8(t - 3) + 2[u(t - 3) - u(t - 4)] + 5(t - 4) - u(t - 5)] 
\end{cases} \]

\[ f(t) = u(t) - u(t - 1) - (t - 4) [u(t - 1) - u(t - 3)] + 4.8(t - 3) + 2u(t - 3) - 2u(t - 4) - (t - 4) u(t - 4) + u(t - 5) + u(t - 5) \]

\[ = u(t) - u(t - 1) - (t - 4) u(t - 1) + 3 u(t - 1) - (t - 3) u(t - 3) - u(t - 3) + 45(t - 3) + 2u(t - 3) - 2u(t - 4) - (t - 4) u(t - 4) + u(t - 5) + u(t - 5) \]

\[ f(t) = u(t) + 2u(t - 1) - (t - 1) u(t - 1) + (t - 3) u(t - 3) + u(t - 3) + 45(t - 3) - (t - 4) u(t - 4) - (t - 5) u(t - 5) \] ... (10.64)

Taking the Laplace transform of eqn 10.64

\[ F(s) = \frac{1}{s} + 2e^{-s} + \frac{1}{s} e^{-3s} + \frac{1}{s} e^{-3s} + \frac{s}{s} \]

\[ = e^{-3s} + \frac{1}{s} \left[ 1 + 2e^{-s} + e^{-3s} - e^{-4s} \right] + \frac{1}{s} \left[ -e^{-s} + e^{-3s} - e^{-4s} + e^{-5s} \right] \] \text{Ans.}
SUMMARY

1. Advantages of Laplace transforms
   (a) It simplifies the operations
   (b) By Laplace transform, the complete solution comes out.
   (c) Linear differential equations can be solved easily.

2. Initial value theorem: If \( f(t) \) and its first derivative are Laplace transformable, then the initial value of \( f(t) \) is
   \[
   f(0^+) = \lim_{t \to 0^+} f(t) = \lim_{s \to \infty} sF(s)
   \]

3. Final value theorem: If \( f(t) \) and its first derivatives are Laplace transformable, then the final value of \( f(t) \) is
   \[
   \lim_{t \to \infty} f(t) = \lim_{s \to 0} sF(s)
   \]

4. Final value theorem does not apply when \( f(t) \) is a periodic function.

5. Table of time functions and their Laplace transform.

<table>
<thead>
<tr>
<th>S.No.</th>
<th>Function ( f(t) )</th>
<th>Waveform</th>
<th>Laplace Transform</th>
<th>Pole Location</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.</td>
<td>( u(t) )</td>
<td>[1]</td>
<td>[\frac{1}{s}]</td>
<td>j0</td>
</tr>
<tr>
<td>2.</td>
<td>( t )</td>
<td>[\frac{1}{s^2}]</td>
<td>[\text{Single pole at } s = 0]</td>
<td>j0</td>
</tr>
<tr>
<td>3.</td>
<td>( t^2 )</td>
<td>[\frac{2}{s^3}]</td>
<td>[\text{Double pole at } s = 0]</td>
<td>j0</td>
</tr>
<tr>
<td>4.</td>
<td>( \sin \omega t )</td>
<td>[\frac{\omega}{s^2 + \omega^2}]</td>
<td>[\text{Triple pole at } s = 0]</td>
<td>j0</td>
</tr>
<tr>
<td>5.</td>
<td>( \cos \omega t )</td>
<td>[\frac{s}{s^2 + \omega^2}]</td>
<td>[\text{Double pole at } s = 0]</td>
<td>j0</td>
</tr>
</tbody>
</table>
Basic Control Actions and Controller Characteristics

11.1. INTRODUCTION
The automatic controller determines the value of controlled variable, compares the actual value to the desired value (reference input), determines the deviation and produces a control signal that will reduce the deviation to zero or to the smallest possible value. The method by which the automatic controller produces the control signal is called mode of control or control action.

The control action may operate through either mechanical, hydraulic, pneumatic or electronic means i.e., controllers can be electrical, hydraulic, pneumatic, electromechanical or electronic types.

11.2. ELEMENTS OF INDUSTRIAL AUTOMATIC CONTROLLER
Fig. 11.1 is a block diagram of an industrial controller.

\[ r - b \]

\[ e = \text{deviation is the difference between controlled variable and set point (reference input)} \]

11.3. CLASSIFICATION OF CONTROLLERS
Controllers are classified depending upon the type of controlling action used. Therefore, they can be classified as
(i) Two-position or ON-OFF controllers.
(ii) Proportional controllers

11.4. TWO POSITION CONTROL
This is also known as ON-OFF control or bang-bang control. This type of controllers is simple and inexpensive and are generally employed on home heating systems, domestic water heaters and industrial control systems.

In this type of control the output of the controller is quickly changed to either a maximum or minimum value depending upon whether the controlled variable (\( b \)) is greater or less than the set point. The minimum value is usually zero.

\[ m = \text{output of the controller} \]
\[ M_1 = \text{Maximum value of output of the controller} \]
\[ M_2 = \text{Minimum value of output of the controller} \]
\[ e = \text{actuating error signal or deviation} \]

The equations for two-position control will be
\[ m = M_1 \quad \text{when } e > 0 \]
\[ m = M_2 \quad \text{when } e < 0 \]

The minimum value \( M_2 \) is usually either zero or \(-M_2\).

Fig. 11.2 shows the block diagram of two position controllers.
can also be defined as the range through which the actuating error signal must move before the switching occurs.

In this type of controller, the control variable always oscillates with a frequency which increases with decreasing width of the dead zone (differential gap). The decrease in dead zone, the number of switching of controller increases. Hence therefore the useful life of the component decreases. Hence-dead band should be designed to prevent the oscillations in ON-OFF controllers.

Two position control mode are used in room air conditioners, heaters, liquid level control in large volume tank.

The example of this type of controller is a room heater. Fig. 11.4 shows the control of temperature. If the temperature drops below the set point, an error signal is produced by the error detector. This error signal energises the relay and the heater turned ON. Similarly, if the temperature increase above a set point, the heater is turned OFF.

**Figure 11.4.**

11.5. PROPORTIONAL CONTROL ACTION

In a controller with proportional control action, there is a continuous linear relation between the output of the controller $m$ (manipulated variable) and actuating error signal $e$ (deviation).

Mathematically

$$m(t) = K_p e(t)$$  \hspace{1cm} (11.4)

or in terms of Laplace transform

$$M(s) = K_p E(s)$$

$$K_p = \frac{M(s)}{E(s)}$$ \hspace{1cm} (11.5)

where $K_p$ is known as proportional gain or proportional sensitivity. The block diagram of proportional controller is shown in Fig. 11.5.

Consider the liquid level system shown in Fig. 11.6. In this system the float lever is directly connected to the control valve. When the level of the liquid raises, the valve close in proportionate amount reduces the inflow to the vessel and vice versa. The inverse of proportional gain or proportional sensitivity is the proportional band and is defined as the change in level necessary to operate the valve through full stroke. Basically proportional controller is an amplifier with adjustable gain.

11.6. INTEGRAL CONTROL ACTION

In a controller with integral control action, the output of the controller is changed at a rate which is proportional to the actuating error signal $e(t)$.

Mathematically,

$$\frac{d}{dt} m(t) = K_i e(t)$$  \hspace{1cm} (11.9)

where $K_i$ is a constant.

Equation 11.6 can also be written as

$$m(t) = K_i \int e(t) + m(0) dt$$  \hspace{1cm} (11.7)

where $m(0)$ control output at $t = 0$.

Laplace transform of Eqn. 11.6

$$M(s) = K_i E(s)$$

or

$$M(s) = \frac{K_i}{s} E(s)$$  \hspace{1cm} (11.8)

Equation 11.8 is the transfer function of integral controller.

The block diagram of integral controller is shown in Fig. 11.7.

The inverse of $K_i$ is called integral time $T_i$ and as defined as the time of change of output caused by a unit change of actuating error signal. The step response of integral controller is shown in Fig. 11.8.

From the Fig. 11.8 it is clear that for positive error, the output of the controller is ramp (positive), for zero error, there is no change in the output of the controller and for negative error, the output of the controller is negative ramp. The integral control action is also known as “Re-set control”.

11.7. DERIVATIVE CONTROL ACTION

In a controller with derivative control action the output of the controller depends on the rate of change of actuating error signal $e(t)$.

Mathematically,

$$m(t) = K_d \frac{d}{dt} e(t)$$  \hspace{1cm} (11.9)

where $K_d$ is known as derivative gain constant

Laplace transform of Eqn. 11.9

$$M(s) = K_d S E(s)$$

or

$$\frac{M(s)}{E(s)} = sK_d$$  \hspace{1cm} (11.10)

Equation 11.10 is the transfer function of the controller. The block diagram is shown in Fig. 11.9.

From equation 11.9 it is clear that when the error is zero or constant, the output of the controller will be zero. Therefore this type of controller cannot be used alone. For this type of controller the gain should be small. The derivative control action also known as rate control.

11.8. PROPORTIONAL-PLUS-INTEGRAL CONTROL ACTION

This is the combination of proportional and integral control action. Mathematically it can be represented by equation 11.11.

$$m(t) = K_p e(t) + K_i \int_0^t e(t) dt$$  \hspace{1cm} (11.11)
or

\[ m(t) = K_p \, e(t) + K_r \, \frac{1}{T_i} \int_0^t e(t) \, dt \]

Laplace transform of Eqn. 11.12

\[ M(s) = K_p \, E(s) + \frac{K_p}{S \, T_i} \, E(s) = E(s) \left[ 1 + \frac{1}{S \, T_i} \right] \]

\[ \frac{M(s)}{E(s)} = K_p \left[ 1 + \frac{1}{S \, T_i} \right] \]

Block diagram is shown in Fig. 11.10.

In equation 11.13 both parameters \( K_p \) and \( T_i \) are adjustable. \( T_i \) is called integral time. The inverse of integral time is called reset rate. Reset rate is defined as the number of times per minute that the proportional part of the response is duplicated. Therefore, Reset rate is also known as "repeats per minute."

Consider the Fig. 11.11. The error varies at \( t = t_1 \). The output of the controller suddenly changes to \( m_p \). Due to proportional action, after that controller output changes linearly with respect to time at a rate \( \frac{K_p}{T_i} \).

For unit step \( (t_1 = 0) \) the response is shown in Fig. 11.12.

From the equation 11.12, it is clear that the proportional sensitivity \( K_p \) affects both the proportional and integral parts of the action.

For unit ramp input. If the actuating error signal is unit ramp then output of the controller is shown in Fig. 11.14.

PD control action reduces the rise time, faster response, improves the bandwidth and improves the damping etc.
11.10. PROPORTIONAL-PLUS-INTEGRAL-PLUS-DERIVATIVE CONTROL ACTION

The combination of proportional, integral and derivative control action is called PID control and the controller is called three action controller. Mathematically

\[ m(t) = K_p \cdot \frac{e(t)}{T_i} + K_p \cdot \frac{1}{T_i} \int_0^t e(t) \, dt + K_p \cdot T_d \cdot \frac{d}{dt} e(t) \]  

Laplace transform

\[ M(s) = K_p \cdot \frac{E(s)}{s} + \frac{K_i}{s} \cdot E(s) + K_p \cdot T_d \cdot SE(s) \]

\[ \frac{M(s)}{E(s)} = K_p \left( 1 + \frac{1}{sT_i} + sT_d \right) \]  

Equation 11.17 is the transfer function.

The block diagram is shown in Fig. 11.16.

\[ K_p \text{ is the proportional gain, } T_i \text{ is the integral time and } T_d \text{ is the derivative time.} \]

Let actuating error signal is given by \( e = At \)

where ‘A’ is a constant and ‘t’ is time.

Put in equation 11.16

\[ m(t) = \frac{K_p}{T_i} \int_0^t At \, dt + K_p \cdot T_d \cdot \frac{d}{dt} At \]

\[ m(t) = K_p \cdot A \left[ t + \frac{t^2}{2T_i} + T_d \right] \]  

From the above equation, the proportional part of the control action repeats the change of error (lower straight line) Fig. 11.17. The derivative part of the control action adds an increment of output so that proportional plus derivative action is shifted ahead in time (middle straight line). The integral part adds a further increment of output proportional to the area under the deviation line. The combination of proportional, integral and derivative action may be made in any sequence.

11.11. RESPONSE WITH P, PI AND PID CONTROLLERS

11.11.1. Proportional Control

The error signal is the difference of reference input and feedback signal. If \( R(s) \) is the input signal and \( E(s) \) is the feedback signal then the error signal is given by \( E(s) = R(s) - B(s) \)

In proportional control the actuating signal is proportional to the error signal \( E(s) \). Therefore it is known as proportional control system. The proportional control action is shown by the block diag (Fig. 11.18).

\[ R(s) = \text{Reference input} \]

\[ E(s) = \text{Error signal} \]

\[ E_a(s) = \text{Actuating signal} \]

\[ C(s) = \text{Output of the system} \]

\[ B(s) = \text{Feedback signal} \]

For quick response, the control system should be underdamped. During the transient period in output the underdamped system has exponentially decaying oscillations. The sluggish (slow moving) underdamped response of a system can be made faster by increasing forward path gain of the system, be steady state error reduced but the maximum overshoot increase.

\[ \frac{E_a(s)}{E(s)} = K_p \]

where, \( K_p \) is known as proportional gain.

11.11.2. PD Controller

A derivative control the actuating signal consists of proportional error signal and derivative of the error signal.

\[ e_a(t) = e(t) + T_d \frac{d}{dt} e(t) \]  

...(11.19)
laplace transform of eqn 11.19
\[ E(s) = E(s) + sT_aE(s) \]
\[ E(s) = E(s) + sT_aE(s) \]
The block diag. of derivative control is shown in fig. 11.19

![Block diagram of derivative control](image)

The overall transfer function of the system shown in fig. 11.19
\[ \frac{C(s)}{R(s)} = \frac{(1+sT_d)}{s(s+2\zeta \omega_n)} \]
\[ = \frac{\omega_n^2}{1+(1+sT_d)} \frac{1}{s} \frac{(1+sT_d)}{s(s+2\zeta \omega_n)} \]
\[ = \frac{\omega_n^2}{s^2 + (2\zeta \omega_n + \omega_n^2 T_d) + \omega_n^2} \]

The characteristic equation of 11.21 is
\[ s^2 + (2\zeta \omega_n + \omega_n^2 T_d) s + \omega_n^2 = 0 \]
Compare the equation 11.22 with
\[ 2\omega_n \zeta' = 2\zeta \omega_n + \omega_n^2 T_d \]
\[ \zeta' = \frac{\xi + \omega_n^2 T_d}{2} \]

Equation (11.23) shows that the damping ratio increase and maximum overshoot is related

Now, the overall transfer function in eqn 11.21 can be written as
\[ \frac{C(s)}{R(s)} = \omega_n^2 T_d \frac{s + 1}{s^2 + 2\zeta \omega_n s + \omega_n^2} \]

The error function is given by,
\[ E(s) = \frac{1}{1 + G(s)H(s)} \]

From fig. 11.19
\[ G(s) = \frac{(1+sT_d)}{s(s+2\zeta \omega_n)} \]
\[ H(s) = 1 \]
\[ \frac{E(s)}{R(s)} = \frac{1}{1 + \frac{\omega_n^2}{s(s+2\zeta \omega_n)}} \]
\[ = \frac{s(s+2\zeta \omega_n)}{s^2 + (2\zeta \omega_n + \omega_n^2 T_d) s + \omega_n^2} \]

Derivative Feedback Control or Rate Feedback Controller

The characteristic equation
\[ s^2 + (2\zeta \omega_n + \omega_n^2 K_t) s + \omega_n^2 = 0 \]
compare with
\[ 2\omega_n \zeta' = 2\zeta \omega_n + \omega_n^2 K_t \]
\[ \zeta' = \frac{\xi + \omega_n^2 K_t}{2} \]

Hence, by using the derivative feedback control, the damping ratio is increased, maximum overshoot reduced.

\[ E(s) = \frac{1}{G(s)H(s) + 1} = \frac{\omega_n^2}{s^2 + (2\zeta \omega_n + \omega_n^2 T_d) s + \omega_n^2} \]

\[ R(s) = \frac{1}{s^2 + (2\zeta \omega_n + \omega_n^2 K_t) s + \omega_n^2} \]

Fig. 11.20
For ramp input

\[ R(s) = \frac{1}{s^2} \]

\[ E(s) = \frac{1}{s^2} \cdot \frac{s^2 + (2\zeta \omega_n + \omega_n^2 K_i) s + \omega_n^2}{s^2 + (2\zeta \omega_n + \omega_n^2 K_i) s + \omega_n^2} \]

Steady state error

\[ e_{ss} = \lim_{s \to \infty} E(s) = \frac{s}{s^2 + (2\zeta \omega_n + \omega_n^2 K_i) s + \omega_n^2} \]

\[ = \frac{1}{s^2} \cdot \frac{s(s + (2\zeta \omega_n + \omega_n^2 K_i))}{s^2 + (2\zeta \omega_n + \omega_n^2 K_i) s + \omega_n^2} \]

\[ = \frac{2\zeta \omega_n + \omega_n^2 K_i}{\omega_n^2} \]

Hence, by using the derivative feedback control the steady state error is increased.

Example 11.1. A closed loop control system with unity feedback is shown in fig. 11.21. By using derivative control the damping ratio is to be made 0.7.

Determine the value of \( T_p \) also determine the rise time, peak time and maximum overshoot without derivative control and with derivative control. The input to the system is unit step.

Solution:

\[ \frac{c(s)}{R(s)} = \frac{14}{s^2 + 1.4s + 14} \]

The characteristic equation \( s^2 + 1.4s + 14 = 0 \) compare with \( s^2 + 2\zeta \omega_n s + \omega_n^2 = 0 \)

\[ 2\zeta \omega_n = 1.4 \quad \text{and} \quad \omega_n = 3.74 \text{ rad/sec.} \]

\[ 2 \times \zeta \times 3.74 = 1.4 \]

\[ \zeta = 0.187 \]

since,

\[ \zeta' = \frac{\zeta + \frac{T_p}{2}}{2} \quad \text{\( T_p = \frac{0.274}{\omega_n} \)} \]

\[ \zeta' = 0.187 + \frac{3.74T_p}{2} \]

\[ C(s) = 0.7 = 0.187 + \frac{3.74T_p}{2} \]

\[ \zeta' = 0.7 \text{ given} \]

\[ T_p = \frac{0.274}{\omega_n} \]

\[ \text{Calculation of rise time, peak time and maximum overshoot without derivative control} \]

\[ t_r = \frac{\pi - \tan^{-1} \sqrt{1-\zeta'^2} \sqrt{\frac{1}{\zeta^2} - \left(\frac{1}{\zeta} - \frac{(0.187)^2}{1}\right)^2}}{\omega_n \sqrt{1-\zeta^2}} = \frac{1.757}{3.67} = 0.478 \text{ sec.} \]
The overall transfer function using derivative feedback control is
\[
\frac{C(s)}{R(s)} = \frac{\omega_n^2}{s^2 + (2\zeta\omega_n + \omega_n^2)K_i + \omega_n^2}
\]

Given that \(\zeta = 0.187\), \(\omega_n = 3.74\) rad/sec, \(K_i = 0.274\)

\[
\frac{C(s)}{R(s)} = \frac{\omega_n^2}{s^2 + 5.23s + 14}
\]

Characteristic eqn \(s^2 + 5.23s + 14 = 0\)

\[2\zeta\omega_n = 5.23, \omega_n = 3.74\]

\(\zeta' = 0.7\)

Rise time \(t_r = \frac{\pi - \tan^{-1}(1 - \zeta'^2)}{\omega_n\sqrt{1 - \zeta'^2}} = \frac{\pi}{3.74\sqrt{1 - 0.7^2}} = 0.877\) sec.

Peak time \(t_p = \frac{\pi}{\omega_n\sqrt{1 - \zeta'^2}} = \frac{\pi}{3.74\sqrt{1 - 0.7^2}} = 1.17\) sec.

Maximum overshoot \(M_p = \frac{\omega_n}{\sqrt{1 - \zeta'^2}} \times 100 = \frac{3.74}{\sqrt{1 - 0.7^2}} \times 100 = 46.4\%

Steady state error \(= \frac{2K_i}{\omega_n + K_i} = \frac{2 \times 0.7}{3.74 + 0.274} = 0.648\)

II.B. PI Controller

In integral control action the actuating signal consists of proportional error signal with integral of the error signal. The block diagram of integral control is shown in fig 11.23.

![Block diagram of integral control](image)

From the block diag. the closed loop transfer function will be
\[
\frac{C(s)}{R(s)} = \frac{1 + \frac{K_i}{s}}{1 + \left(1 + \frac{K_i}{s}\right)\frac{\omega_n^2}{s(\omega_n^2 + 2\zeta\omega_n k) + \omega_n^2 k}}
\]

... (11.31)

The characteristic equation is the third order equation hence the system becomes third order system.

\[
\frac{E(s)}{R(s)} = \frac{1 + G(s)}{H(s)} = \frac{1}{1 + \left(1 + \frac{K_i}{s}\right)\frac{\omega_n^2}{s(\omega_n^2 + 2\zeta\omega_n k) + \omega_n^2 k}}
\]

\[
\frac{E(s)}{R(s)} = \frac{1}{s^3 + (2\zeta\omega_n + \omega_n^2 + K_i\omega_n^2)
\]

Example 11.2: In Example 11.1 it is desired the damping ratio be 0.7. Determine the derivative rate feedback \(K_i\). Also determine the peak time, rise time, steady state error and max. overshoot with and without feedback for unit ramp input.

Solution: The characteristic eqn. \(s^2 + 1.4s + 14 = 0\)

\[\zeta = 0.187 \& \omega_n = 3.74\) rad/sec.

\[\zeta' = \frac{\zeta + \frac{\omega_n}{\sqrt{2}}}{2} = \frac{0.187 + 3.74K}{2} \Rightarrow K = 0.274\]

The calculation for rise time, peak time and max. overshoot without feedback is same as in previous solution. The steady state error without feedback.

\[\varepsilon_{ss} = \frac{2K_i}{\omega_n} = \frac{2 \times 0.187}{3.74} = 0.1\]
\[ E(s) = R(s) \frac{s^2 + 2s\zeta\omega_n + \omega_n^2}{s^3 + 2s\zeta\omega_n s + \omega_n^2 s + \omega_n^2 + K_i\omega_n^2} \]

From eqn. (11.32) if \( R(s) = \frac{1}{s^2} \) then the steady state error will be zero.

If the input is parabolic input i.e. \( R(s) = \frac{1}{s} \) then we have steady state error.

\[ E(s) = \frac{1}{s^3} \frac{s^2 + 2s\zeta\omega_n + \omega_n^2}{s^3 + 2s\zeta\omega_n s + \omega_n^2 s + \omega_n^2 + K_i\omega_n^2} \]

\[ e_{ss} = \lim_{s \to 0} E(s) = \frac{2\zeta\omega_n}{K_i\omega_n} = \frac{2\zeta}{\omega_n} \]

11.11.5. Proportional Plus Derivative Plus Integral Control (PID)

In PI D control the actuating signal consists of proportional error signal added with derivative and integral of error signal.

\[ \frac{R(s)}{E(s)} = \frac{1}{s^2 + 2s\zeta\omega_n + \omega_n^2} \]

\[ E_a(s) = E(s) + sT_d E(s) = \frac{K_i}{s} E(s) = E(s) \left[ 1 + sT_d + \frac{K_i}{s} \right] \]

The block diaq of PI D control is shwon in fig. 11.24.

\[ E(s) = \frac{10}{s^3 + 2s + 10} \]

The overall transfer function \( \frac{C(s)}{R(s)} \) will be

\[ \frac{C(s)}{R(s)} = \frac{10}{s(s + 2)} \]

\[ \frac{C(s)}{R(s)} = \frac{10}{s(s + 2)} \]

Characteristic equation \( s^3 + 2s + 10 = 0 \)

\[ \omega_n^2 = 10 \quad \omega_n = 3.16 \text{ rad/sec.} \]

\[ 2\zeta\omega_n = 2 \quad \zeta = 0.316 \]

Maximum overshoot \( M_p = e^{-\sqrt{\frac{4}{\zeta}} - 1} \times 100 \)

\[ M_p = e^{0.5 - 0.316} \times 100 = 16.3\% \]

Settling time \( t_s = \frac{4}{\zeta\omega_n} = 4 \times 0.5 \times 3.16 = 2.53 \text{ sec.} \]

Steady state error \( e_{ss} = \lim_{s \to 0} sE(s) \)

\[ E(s) = \frac{1}{s} \frac{1}{s^2 + 2s + 10} \]

\[ e_{ss} = \lim_{s \to 0} sE(s) = \frac{1}{s^2 + 2s + 10} = \frac{1}{1 + G(s)} \]

Characteristic eqn \( \frac{s^2 + 2s + 10}{s(s + 2)} \)

\[ \frac{s^2 + 2s + 10}{s(s + 2)} \]

Comparison \( s^2 + 2s + 10 \)

\[ \omega_n^2 = 10 \quad \omega_n = 3.16 \text{ rad/sec.} \]

\[ 2\zeta\omega_n = 2 \quad \zeta = 0.316 \]

\[ M_p = e^{-\sqrt{\frac{4}{\zeta}} - 1} \times 100 = 35.14\% \]

\[ t_s = \frac{4}{\zeta\omega_n} = 4 \times 0.316 \times 3.16 = 4 \text{ sec.} \]

Example 11.4. A unity feedback system is characterised by an open loop transfer function

\[ G(s) = \frac{K}{s(s + 10)} \]

Determine the gain \( K \) so that the system will have a damping ratio of 0.5. For this value of \( K \) determine setting time, peak overshoot and peak time for a unit step input.

Solution: The characteristic eqn \( \frac{s^2 + 2s + 10}{s^2 + 2s + 10} \)

\[ \frac{s^2 + 2s + 10}{s^2 + 2s + 10} \]

Comparison \( s^2 + 2s + 10 \)

\[ \omega_n^2 = K \quad \omega_n = \sqrt{K} \]

\[ 2\zeta\omega_n = 10 \]

\[ \omega_n = \sqrt{10} = 10 \text{ rad/sec.} \]
Example 11.5. A feedback system employing output-rate damping, shown in fig. 11.26. (a) In the absence of derivative feedback ($K_D = 0$), determine the damping factor and natural frequency of the system. What is the steady state error resulting from unit ramp input?

(b) Determine the derivative feedback constant $K_D$ which will increase the damping factor of the system to 0.6. What is the steady state error resulting from unit ramp input with this setting of the derivative feedback constant?

\[ G(s) = \frac{10}{s(s+2)} \]

characteristic equation $s^2 + 2\zeta\omega_n s + \omega_n^2 = 0$

\[ \omega_n^2 = 10 \quad \text{and} \quad \omega_n = 3.16 \text{ rad/sec} \]

2. $s^2 + 2\zeta\omega_n s + \omega_n^2 = 0$

or, $s^2 + 2s + 10 = 0$ compare with $s^2 + 2\zeta\omega_n s + \omega_n^2 = 0$

\[ \omega_n^2 = 10 \quad \text{and} \quad \omega_n = 3.16 \text{ rad/sec} \]

\[ 2\zeta\omega_n = 2 \quad \text{or} \quad \zeta = 0.316 \]

Steady state error $e_{ss} = \frac{2\zeta}{\omega_n} = \frac{2 \times 0.316}{3.16} = 0.2 \text{ rad}$

Solution: (a) When $K_D = 0$

\[ \frac{G(s)}{H(s)} = \frac{10}{s(s+2)} \]

or, $s^2 + 2s + 10 = 0$ compare with $s^2 + 2\zeta\omega_n s + \omega_n^2 = 0$

\[ \omega_n^2 = 10 \quad \text{and} \quad \omega_n = 3.16 \text{ rad/sec} \]

2. $s^2 + 2\zeta\omega_n s + \omega_n^2 = 0$

\[ \zeta = 0.316 \]

Steady state error $e_{ss} = \frac{2\zeta}{\omega_n} = \frac{2 \times 0.316}{3.16} = 0.2 \text{ rad}$

(b) The overall transfer function of fig. 11.26.

\[ \frac{C(s)}{R(s)} = \frac{10}{s^2 + 2s + 10} \]

The characteristic equation $s^2 + 2s + 10 = 0$ compare with $s^2 + 2\zeta\omega_n s + \omega_n^2 = 0$

\[ \omega_n^2 = 10 \quad \text{and} \quad \omega_n = 3.16 \text{ rad/sec} \]

\[ 2\zeta\omega_n = 2 \quad \text{or} \quad \zeta = 0.316 \]

Steady state error $e_{ss} = \frac{2\zeta}{\omega_n} = \frac{2 \times 0.316}{3.16} = 0.2 \text{ rad}$

Example 11.6. Block diag. model of a position control system shown in fig. 11.27.

(a) In the absence of derivative feedback ($K_D = 0$) determine the damping ratio of the system for amplifier gain $K_A = 5$. Also find the steady state error to unit ramp input.

(b) Find suitable values of the parameters $K_i$ and $K_A$ so that the damping ratio of the system is increased to 0.7 without affecting the steady state error as obtained in part (a). (GATE 1992)

\[ e_{ss} = \lim_{s \to 0} \frac{s}{1 + G(s)H(s)} = \frac{1}{1 + \frac{s}{s(1 + 0.5s + 1.8s + 1.8s + 10)} \times 100} = 16.3% \]

\[ e_{ss} = \lim_{s \to 0} \frac{s}{s(1 + 0.5s + 1.8s + 10)} \times 100 = 0.38 \text{ rad} \]

The characteristic equation $s^2 + 2s + 10 = 0$ compare with $s^2 + 2\zeta\omega_n s + \omega_n^2 = 0$

\[ \omega_n^2 = 10 \quad \text{and} \quad \omega_n = 3.16 \text{ rad/sec} \]

\[ 2\zeta\omega_n = 2 \quad \text{or} \quad \zeta = 0.316 \]

Steady state error $e_{ss} = \frac{2\zeta}{\omega_n} = \frac{2 \times 0.316}{3.16} = 0.2 \text{ rad}$

Solution:

(a) The solution of part 'a' is same as in problem 11.5

\[ \zeta = 0.316 \text{ and } e_{ss} = 0.2 \text{ rad} \]

(b) The overall transfer function of fig. 11.27

\[ \frac{C(s)}{R(s)} = \frac{2K_A}{s^2 + s(2 + 2K_i) + 2K_A} \]

The characteristic equation $s^2 + s(2 + 2K_i) + 2K_A = 0$ compare with $s^2 + 2\zeta\omega_n s + \omega_n^2 = 0$

\[ \omega_n^2 = 10 \quad \text{and} \quad \omega_n = 3.16 \text{ rad/sec} \]

\[ 2\zeta\omega_n = 2 \quad \text{or} \quad \zeta = 0.316 \]

Steady state error $e_{ss} = \lim_{s \to 0} sE(s)$

\[ e_{ss} = \lim_{s \to 0} s \cdot R(s) = \frac{1}{1 + G(s)H(s)} = \frac{1}{1 + \frac{s}{s(1 + 0.5s + 1.8s + 1.8s + 10)}} \times 100 = 16.3\% \]

\[ e_{ss} = \lim_{s \to 0} s \cdot R(s) = \frac{1}{1 + \frac{s}{s(1 + 0.5s + 1.8s + 1.8s + 10)}} \times 100 = 0.38 \text{ rad} \]

\[ e_{ss} = \lim_{s \to 0} s \cdot R(s) = \frac{1}{1 + \frac{s}{s(1 + 0.5s + 1.8s + 1.8s + 10)}} \times 100 = 16.3\% \]

\[ e_{ss} = \lim_{s \to 0} s \cdot R(s) = \frac{1}{1 + \frac{s}{s(1 + 0.5s + 1.8s + 1.8s + 10)}} \times 100 = 0.38 \text{ rad} \]

\[ e_{ss} = \lim_{s \to 0} s \cdot R(s) = \frac{1}{1 + \frac{s}{s(1 + 0.5s + 1.8s + 1.8s + 10)}} \times 100 = 16.3\% \]

\[ e_{ss} = \lim_{s \to 0} s \cdot R(s) = \frac{1}{1 + \frac{s}{s(1 + 0.5s + 1.8s + 1.8s + 10)}} \times 100 = 0.38 \text{ rad} \]
11.12. ELECTRONIC CONTROLLERS

11.12.1. Proportional Controller

\[ e_o = K (e_l - e_f) \]
\[ e_f = e_o \frac{R_2}{R_1} \]

Laplace transform of (11.34) and (11.35)
\[ E_o(s) = K [E_l(s) - E_f(s)] \]
\[ E_f(s) = E_o(s) \frac{R_2}{R_1} \]

From equations (11.36) and (11.37)
\[ E_o(s) = K \left[ \frac{E_l(s) - E_o(s)}{R_2} \right] \]
\[ \frac{E_o(s)}{E_l(s)} = \frac{R_2}{R_1} \]

Put
\[ \frac{R_1}{R_2} = K_p \]
\[ \frac{E_o(s)}{E_o(s)} = K_p \]

11.12.2. PD Controller

\[ e_o = K (e_l - e_f) \]
\[ e_f = \frac{R}{C} \int e_o dt \]

Laplace transform of (11.39) and (11.40)
\[ E_o(s) = K [E_l(s) - E_f(s)] \]
\[ E_f(s) = R l(s) + \frac{1}{SC} l(s) \]

\[ E_f(s) = \frac{1}{C} \int l dt \]

Laplace transform of (11.43)
\[ E_f(s) = \frac{1}{sC} l(s) \]

From (11.42) and (11.44)
\[ \frac{E_f(s)}{E_o(s)} = \frac{1}{1 + RCs} \]

Put the value of \( E_f(s) \) from (11.45) in (11.41) and then solve for
\[ \frac{E_o(s)}{E_l(s)} = \frac{K}{1 + \frac{K}{1 + RCs}} \]

If \( \frac{1}{1 + RC} \gg 1 \)
then
\[ \frac{E_o(s)}{E_l(s)} = 1 + RCS \]

or
\[ \frac{E_o(s)}{E_l(s)} = 1 + T_d S \]

where
\[ T_d = RC \]

Thus, we get PD type control action.

11.12.3. PI Controller
Laplace transform of (11.48)
\[ E_d(s) = k[E_i(s) - E_f(s)] \]
\[ e_i = Ri + \frac{1}{C} \int e(t) \, dt \]

Laplace transform of (11.50)
\[ E_f(s) = Rl(i(s) + \frac{1}{sc} l(t)) \]
\[ E_i(s) = l(s) \left[ \frac{RCs + 1}{Sc} \right] \]
\[ E_r = Ri \]

Laplace transform of (11.52)
\[ E_f(s) = RI(s) \]

From (11.51) and (11.53)
\[ \frac{E_r(s)}{E_i(s)} = \frac{RCs}{1 + RCS} \]

From (11.54) put \( E_i \) in (11.49) and solve for \( \frac{E_d(s)}{E_i(s)} = \frac{K}{1 + \frac{KRCS}{1 + RCS}} \)

If \( \frac{KRC}{1 + RCS} \gg 1 \) then
\[ \frac{E_d(s)}{E_f(s)} = \frac{1 + RCS}{RCs} = \frac{1}{RCS} + 1 \]
\[ \frac{E_r(s)}{E_i(s)} = \frac{1}{RCS} + 1 \]

where, \( T_r = RC \)
Thus, we get PI control action.

**11.24. PID Controller**

![PID Controller Diagram](Fig. 11.31)
11.13. EFFECT OF INTEGRAL AND DERIVATIVE CONTROL ON THE SYSTEM PERFORMANCE

11.13.1. Integral Control Action

In proportional control of a plant whose transfer function does not possess an integrator \( 1/s \), there is a steady-state error or an offset in the response to a step input. Such an offset can be eliminated if the integral control action is introduced in the controller.

The integral control action will removing offset or steady-state error may lead to oscillatory response of slowly decreasing amplitude or sometimes even increasing amplitude, both of which are usually undesirable.

11.13.2. Derivative Control Action

Derivative control action when added to proportional controller provides a means of obtaining a controller with high sensitivity and advantage of using derivative control action is that it responds to rate of change of actuating error and produces an efficient correction before the magnitude of actuating error becomes too large. Derivative control thus anticipate the actuating error, initiate an easily corrective action and leads to increase the stability of the system. Although derivative control does not effect the steady-state error directly, it adds damping to the system and thus permits the use of a larger value of gain \( K \) which will result in an improvement in the steady-state accuracy. Because derivative control operates on the rate of change of actuating error, not as the actuating error itself, this mode is never used alone. It is always used in combination with proportional or proportional integral control.

**SUMMARY**

A controller accepts the error and made proper corrective action. The output of the controller is then applied to the control element.

**EXERCISE**

1. What are the basic elements of an industrial automatic controller?
   - Explain Two position or ON-OFF control.
3. Show that the steady state error increase by using the derivative feedback control.
4. Solve the example 11.1 for damping ratio 0.8.

**SEMI-OBJECTIVE TYPE QUESTIONS**

(i) Define mode of control.
(ii) With a suitable diagram show the elements of industrial automatic controller.
(iii) Classify the various type of controllers.
(iv) Write short note on two position control.
(v) Write short note on proportional control action.
(vi) Discuss in brief derivative control action.
(vii) What is PID controller?
(viii) Write short note on derivative and Integral control action.
(ix) Write short note on PD control action.
(x) Write short note on PI control action.
(xi) Write short note on PID control action.

(Article 11.1)
(Article 11.2)
(Article 11.3)
(Article 11.4)
(Article 11.5)
(Article 11.6)
(Article 11.7)
(Article 11.10)
(Summary)
(Summary)
Chapter 12

Miscellaneous Problems

Example 12.1. Draw the electrical analogous circuit of the system shown in fig. 1.48 using $v$ and $f-i$ analogy and also write the equations.

**Solution:** $F-V$ analogy:

\[
V(s) = q_1 \frac{1}{C_1} + s^2 L_1 q_1 + \frac{1}{C_2} (q_1 - q_2)
\]

\[
\frac{1}{C_2} (q_2 - q_1) + s^2 L_2 q_2 + \frac{1}{C_3} q_2 = 0
\]

$F-i$ Analogy

\[
i(s) = \frac{1}{L_1} \phi_1 + s^2 C_1 \phi_1 + \frac{1}{L_2} (\phi_1 - \phi_2)
\]

\[
\frac{1}{L_2} (\phi_2 - \phi_1) + s^2 C_2 \phi_2 + \frac{1}{L_3} \phi_2 = 0
\]

Example 12.2. Draw the mechanical equivalent network of the system shown in fig. 1.50.

Example 12.3. Draw the mechanical equivalent network of the system shown in fig. 1.53.

Example 12.4. Draw the mechanical equivalent network of the system shown in fig. Also draw the electrical analogous circuit and write the set of equation using $f-i$ and $f-i$ analogy.
Example 12.5. Draw the mechanical equivalent network of the following rotational system and write the system equations.

Solution: In the given fig the torque is applied to the spring so, there are two displacements at the ends of the spring, \(J_1\) is under \(\theta_2\), \(K_2\) and \(B_2\) are in parallel between \(\dot{\theta}_2(t)\) and \(\theta_2(t)\), \(J_2\) and \(K_3\) are under \(\dot{\theta}_3\).

\[
T(t) = K_1 (\dot{\theta}_1 - \dot{\theta}_2) \\
K_1 (\theta_2 - \theta_1) + J_1 \frac{d^2 \theta_2}{dt^2} + K_2 (\theta_2 - \theta_3) + B_2 \frac{d \theta_2}{dt} (\theta_2 - \theta_3) = 0 \\
J_2 \frac{d^2 \theta_3}{dt^2} + K_3 (\theta_3 - \theta_2) + B_2 \frac{d \theta_3}{dt} (\theta_3 - \theta_2) = 0
\]

Example 12.6. For Example (12.5) draw the electrical analogous circuit using T-V analogy.
Example 12.8. Draw the mechanical equivalent network of the given system and also draw the electrical analogous circuit using $f-v$ analogy.

Solution: Mechanical equivalent network.

Since the force is directly applied to the spring, it will store the energy. The spring $K_1$ is under $x_1$ and $x_2$, mass $M_1$ is under $x_1$, $K_2$ is under $x_2$ and $x_3$, $M_2$ is under $x_3$ and spring $K_3$ is also under $x_3$.

- At node $x_1$: $F = K_1(x_1 - x_2)$
- At node $x_2$: $K_1(x_2 - x_1) + M_1 \frac{d^2 x_2}{dt^2} + K_2(x_2 - x_3) = 0$
- At node $x_3$: $K_2(x_3 - x_2) + M_2 \frac{d^2 x_3}{dt^2} = K_3 x_3 = 0$

Example 12.9. Draw the mechanical equivalent circuit and write the system equations. Also draw the electrical analogous circuit using $f-v$ analogy.

Example 12.10. Draw the mechanical network of given system and draw the electrical analogous circuit using $f-v$ and $f-i$ analogy.
Example 12.12. Draw the mechanical equivalent circuit and write the system equations.

Solution:

Example 12.13. For above question no. 12.12 draw f-v and f-i analogy.

Example 12.14. Write the differential equations governing the behaviour of the mechanical system shown in fig. Also obtain the analogous electrical circuit based on f-v and f-i analogy.

Solution: Mechanical equivalent network.
Example 12.15. Find the transfer function \( \frac{X(s)}{E(s)} \) for the electromechanical system shown in Fig. 12.31.

Solution: The given system consists of two parts (a) electrical (b) mechanical.

First consider the electrical system.

Apply KVL in both meshes:

\[
E(t) = R_1i_1(t) + \frac{1}{C} [i_1(t) - i_2(t)]
\]

\[
0 = \frac{1}{sC} [i_2(t) - i_1(t)] + L \frac{di_2(t)}{dt} + \varepsilon_b
\]

Laplace transform of both equations

\[
E(s) = R_1i_1(s) + \frac{1}{sC} [i_1(s) - i_2(s)]
\]

\[
0 = \frac{1}{sC} [I_2(s) - I_1(s)] + \varepsilon_b
\]

Now consider the mechanical system:

\[
F(t) = M \frac{d^2x(t)}{dt^2} + f \frac{dx(t)}{dt} + 2kx(t)
\]

\[
F(s) = s^2 MX(s) + sfX(s) + 2kX(s)
\]

From mechanical network

\[
E(t) = K_2I_2(t)
\]

Put the value of \( F(s) \) from (6) in (7) we get

\[
K_2I_2(s) = X(s) [s^2 M + sf + 2k]
\]

From equation (5) put the value of \( E_2(s) \) in (4)

\[
0 = \frac{1}{sC} I_2(s) - \frac{1}{sC} I_1(s) + sL I_2(s) + K_3 \varepsilon_b X(s)
\]

or

\[
\frac{1}{sC} I_2(s) = \frac{1}{sC} I_2(s) + sL I_2(s) + K_3 \varepsilon_b X(s)
\]

Put the value of \( I_1(s) \) from equation (9) in equation (3)

\[
E(s) = R_1I_1(s) + \frac{1}{sC} I_2(s) - \frac{1}{sC} I_2(s)
\]

\[
E(s) = \left[ R_1 + \frac{1}{sC} \right] I_2(s) + s^2 R_2 L C I_2(s) + s^2 \left( C K_1 X(s) + \frac{1}{sC} I_2(s) \right)
\]

\[
E(s) = R_1 I_2(s) + s^2 R_2 L C I_2(s) + s^2 R_2 K_1 X(s) + \frac{1}{sC} I_2(s) + sL I_2(s) + sK_1 X(s) - \frac{1}{sC} I_2(s)
\]

\[
E(s) = I_2(s) [R_1 + s^2 R_2 L C + sL] + X(s) s K_1 [1 + R_1 C s]
\]

\[
I_2(s) = \frac{E(s) - X(s) s K_1 (1 + R_1 C s)}{R_1 + s^2 R_2 L C + sL}
\]
Put the value of \( I_2(s) \) from equation (10) in equation (8)

\[
K_2 \left[ \frac{E(s) - X(s)}{s K_1 (1 + R_1 C_s)} \right] = X(s) \left[ s^2 M + s f + 2K \right]
\]

\[
K_2 \frac{E(s) - X(s)}{s K_1 (1 + R_1 C_s)} = X(s) \left[ s^2 R_1 M + s R_1 f + 2K R_1 \right.
\]

\[+ s^2 M R_1 C + s^3 R_1 L C f + 2 s^2 R_1 L C K + s^2 M L + s^2 R L + 2 s R \]

\[
K_2 \frac{E(s) - X(s)}{s K_1 (1 + R_1 C_s)} = X(s) \left[ s^4 R_1 L C M + s^3 L (R_1 C f + M) + s^2 R_1 M + s^2 R_1 f + 2 s R_1 L C K \right.
\]

\[+ s^2 M R_1 C + s^3 R_1 L C f + 2 s^2 R_1 L C K + s^2 M L + s^2 R L + 2 s R \]

\[
= X(s) \left[ s^4 R_1 L C M + s^3 L (R_1 C f + M) + s^2 (R_1 M + f L + 2 R_1 L C K \right.
\]

\[+ s^2 R_1 L C f + 2 s^2 R_1 L C K + s^2 M L + s^2 R L + 2 s \]

\[
= X(s) \left[ s^4 R_1 L C M + s^3 L (R_1 C f + M) + (R_1 M + f L + 2 R_1 L C K \right.
\]

\[+ s^2 R_1 L C f + 2 s^2 R_1 L C K + s^2 M L + s^2 R L + 2 s \]

\[
X(s) \left[ \frac{1}{s^2 R_1 L C M + s^3 L (R_1 C f + M) + s^2 (R_1 M + f L + 2 R_1 L C K \right.
\]

\[+ s^2 R_1 L C f + 2 s^2 R_1 L C K + s^2 M L + s^2 R L + 2 s \]

\[
= X(s) \left[ \frac{1}{s^2 R_1 L C M + s^3 L (R_1 C f + M) + s^2 (R_1 M + f L + 2 R_1 L C K \right.
\]

\[+ s^2 R_1 L C f + 2 s^2 R_1 L C K + s^2 M L + s^2 R L + 2 s \]

\[
= X(s) \left[ \frac{1}{s^2 R_1 L C M + s^3 L (R_1 C f + M) + s^2 (R_1 M + f L + 2 R_1 L C K \right.
\]

\[+ s^2 R_1 L C f + 2 s^2 R_1 L C K + s^2 M L + s^2 R L + 2 s \]

This is the required answer.

**Example 12.16.** For mass-damper spring combination shown in fig.12.34 draw the mech. Equivalent network, write the system equations and draw \( f \)-\( v \), \( f \)-\( i \) analog.

**Solution:** Mechanical network (Fig. 12.35)

at node \( x \):

\[ F = K(x - y) \]

\[ 0 = K(y - x) + M \frac{dy}{dt} + B \frac{dy}{dt} \]

\( f \)-\( v \) analogy


![Fig. 12.34](image)


![Fig. 12.35](image)

\( f \)-\( v \) analogy


![Fig. 12.36](image)

\( f \)-\( v \) analogy


![Fig. 12.37](image)

\( f \)-\( v \) analogy


![Fig. 12.38](image)

\( f \)-\( v \) analogy


![Fig. 12.39](image)

\( f \)-\( v \) analogy

**Solution:** For the mechanical system shown in fig.12.38(a) draw mechanical equivalent network (b) write the differential equations (c) draw the electrical analogous circuit using \( f \)-\( v \) and \( f \)-\( i \) analog.

**Example 12.17.** For the mechanical system shown in fig.12.38(a) draw mechanical equivalent network (b) write the differential equations (c) draw the electrical analogous circuit using \( f \)-\( v \) and \( f \)-\( i \) analog.
Example 12.18. For the given mechanical system draw the mechanical equivalent network and write the system equations.

\[ F(t) = K_1 (x - y) \]

Solution: Mechanical equivalent network (Fig. 12.42)

at node \( x \):
\[ V(y) = K_2 y + B \frac{d}{dt} y = 0 \]

Example 12.19. For the system shown in question (12.18) draw the electrical analogous circuit using \( f-v \) and \( f-i \) analogy.

For \( f-v \) analogy:
\[ V = \frac{1}{C_1} (q_1 - q_2) \]

Since
\[ i = \frac{dq}{dt} \]

or,
\[ I(s) = sQ(s) \]

For \( f-i \) analogy:
\[ V = \frac{1}{L_1} \left( I_1(s) - I_2(s) \right) \]

\[ 0 = \frac{1}{C_2} \left[ I_2(s) - I_1(s) \right] + \frac{1}{sC_2} I_2(s) + RL_2(s) \]

Example 12.20. For the given mechanical system draw (a) mechanical equivalent network (b) electrical analogous circuit (c) write the system equations.

\[ M_1 \]
\[ M_2 \]
\[ M_3 \]

\[ F = \cos \Theta \]
Solution: Mechanical equivalent network

\[ F = M_2 \frac{d^2 x_2}{dt^2} + K_3 (x_2 - x_3) \]

at node \( x_2 \):
\[ K_3 (x_2 - x_1) + M_2 \frac{d^2 x_2}{dt^2} + K_2 (x_2 - x_3) = 0 \]

at node \( x_3 \):
\[ K_3 (x_3 - x_2) + M_1 \frac{d^2 x_3}{dt^2} + b_1 \frac{dx_3}{dt} + K_1 x_3 = 0 \]

Laplace transform of above equations

\[ F = f \cos \omega t \text{ given} \]

\[ F(s) = \frac{F_0}{s^2 + \omega^2} \]

\[ \frac{X_1(s)}{s^2 + \omega^2} = s^2 M_2 X_2(s) + K_3 [X_1(s) - X_2(s)] \]

\[ K_2 [X_2(s) + X_1(s)] + s^2 M_2 X_2(s) + K_3 X_3(s) - X_3(s) = 0 \]

\[ K_3 [X_3(s) - X_2(s)] + s^2 M_1 X_3(s) + s b_1 X_3(s) + K_1 X_3(s) = 0 \]

F-V analogy:

\[ V(s) = s^2 L_3 \phi_1(s) + \frac{1}{C_3} [\phi_1(s) - \phi_2(s)] \]

\[ \frac{1}{C_3} [\phi_2(s) - \phi_1(s)] + s^2 L_2 \phi_2(s) + \frac{1}{C_2} [\phi_2(s) - \phi_3(s)] = 0 \]

\[ \frac{1}{C_2} [\phi_3(s) - \phi_2(s)] + s^2 L_4 \phi_3(s) + s b_1 \phi_3(s) + \frac{1}{C_1} \phi_3(s) = 0 \]

Since
\[ j = \frac{dx}{dt} \]
\[ I(s) = s q(s) \quad \text{or} \quad q(s) = \frac{I(s)}{s} \]

These are the required equations of the given system.

Example 12.21. Using block diagram reduction technique, find closed loop transfer function of the system whose block diagram is given in fig. 12.48 (a) when \( R_1 = 0 \) (b) when \( R_2 = 0 \).
Solution: When $R_2 = 0$

Fig. 12.48(a)

New remove the summing point since feedback is negative therefore put sign with $H_3$

Fig. 12.49

Fig. 12.49(a)

Fig. 12.49(b)
Example 12.22. Determine $\frac{C(s)}{R(s)}$ by block reduction method.

\[ \frac{C(s)}{R(s)} = \frac{G_3}{1 + G_1 G_2 G_3 (H_1 + H_2)} \]

Ans.

\[ \frac{C(s)}{R(s)} = \frac{G_1 G_2 G_3}{1 + G_1 G_2 G_3 (H_1 + H_2) + G_1 G_2 G_3 H_1 H_2 H_3} \]
Example 12.23: Using block reduction technique, find the closed loop transfer function.

Solution: Redraw the diagram

Example 12.24: Find $c(s)/R(s)$.

Solution: Block are in parallel

$C(s) = \frac{G_1 + G_2}{1 + (G_1 + G_2)(G_1 - G_2)}$
Example 12.25. From the block diagram find $\frac{C_1}{R_2}$ assuming $R_1 = 0$

Solution: When $R_1 = 0$

Example 12.26. Alternative method to determine $C/R_1$ and other ratio for example 1.34.
Example 12.27. Find the transfer function for the signal flow graph shown in fig. 3.10b.

Solution: Forward path:

\[ s_1 = \frac{1}{s} \left( \frac{1}{s} \right) = \frac{1}{s^2} \quad \Delta_1 = 1 \]

Individual loops:
\[ L_1 = -\frac{3}{s} \]
\[ L_2 = \frac{5}{s^2} \]
\[ L_3 = \frac{1}{s^2} \]

\[ \Delta = 1 - (L_1 + L_2 + L_3) = 1 + \left[ \frac{3}{s} + \frac{5}{s^2} + \frac{2}{s^3} \right] \]
\[ \Delta = 1 + \left[ \frac{3s^2 + 5s + 2}{s^3} \right] = \frac{s^3 + 3s^2 + 5s + 2}{s^3} \]

\[ \frac{C(s)}{R(s)} = \frac{s_1}{s} \Delta = \frac{1}{s^3 + 3s^2 + 5s + 2} \]

Ans.

Example 12.28. For example 12.7 draw the signal flow graph and find the transfer function.

Solution:

\[ R(s) \]
\[ 1 \]
\[ C(s) \]

Fig. 12.55.

Forward paths:
\[ s_1 = C_1 G_2 G_3 \quad \Delta_1 = 1 \]
\[ s_2 = C_1 G_4 \quad \Delta_2 = 1 \]

Individual loops:
\[ L_1 = -C_1 G_2 H_1 \quad L_2 = -C_2 G_3 H_2 \]
\[ L_3 = -C_1 G_3 H_3 \quad L_4 = -C_4 H_2 \]

Two non-touching loops : \[ L_1, L_3 \]

\[ \Delta = 1 - (L_1 + L_2 + L_3) + [L_1 L_3] \]
\[ = 1 + \left[ \frac{R_3 R_4}{R_1 R_2} + \frac{R_4 R_3}{R_1 R_2} \right] \]
\[ = 1 + \frac{R_3 R_4 + R_2 R_1 + R_4 R_3 + R_3 R_4}{R_1 R_2} \]

\[ \frac{C(s)}{R(s)} = \frac{s_1}{s} \Delta = \frac{R_1 R_3}{R_1 R_2 + R_2 R_1 + R_4 R_3 + R_3 R_4 + R_3 R_4} \]

Ans.

Example 12.29. Find the transfer function of the electrical circuit whose SFG is shown in fig.

Solution: Forward path:
\[ s_1 = \frac{1}{R_1} R_3 \frac{1}{R_2} R_4 \quad \Delta_1 = 1 \]

Individual loops:
\[ L_1 = \frac{R_3}{R_1} \]
\[ L_2 = \frac{R_3}{R_2} \]
\[ L_3 = \frac{R_4}{R_2} \]

Two non-touching loops : \[ L_1, L_3 \]

\[ \Delta = 1 - (L_1 + L_2 + L_3) + [L_1 L_3] \]
\[ = 1 + \left[ \frac{R_3}{R_1} + \frac{R_3}{R_2} + \frac{R_4}{R_2} + \frac{R_4}{R_2} \right] \]
\[ = 1 + \frac{R_1 R_3 + R_2 R_1 + R_3 R_4 + R_3 R_4}{R_1 R_2} \]

\[ \frac{C(s)}{R(s)} = \frac{s_1}{s} \Delta = \frac{R_1 R_3}{R_1 R_2 + R_2 R_1 + R_3 R_4 + R_3 R_4 + R_3 R_4} \]

Ans.

Example 12.30. For the given electrical network draw the signal flow graph and find the transfer function by mason gain formula.

Fig. 12.56.
Solution: Draw the transform network

\[ l_1(s) = \frac{V_1(s) - V_i(s)}{R_1 + sL_1} \]

\[ V_i(s) = \frac{1}{sC}[l_1(s) - l_2(s)] \]

\[ l_2(s) = \frac{V_1(s) - V_i(s)}{sL_2} \]

\[ V_0(s) = \frac{R_2}{sL_2} l_2(s) \]

\[ R(s) \quad V(s) \quad 1/sC \quad \frac{1}{sL_2} \quad V_i(s) \quad 1/sC \quad V_1(s) \quad l_1(s) \quad 1/sC \quad l_2(s) \quad 1/sL_2 \quad l_3(s) \quad C(s) \]

Forward path:

\[ s_1 = \frac{1}{R_1 + sL_1} \quad s_2 = \frac{1}{R_2 + sL_2} \]

\[ s_1 = \frac{1}{sC(R_1 + sL_1)} \quad s_2 = \frac{1}{sL_2} \quad \Delta_1 = 1 - 0 = 1 \]

Individual loops:

\[ L_1 = \frac{1}{sC(R_1 + sL_1)} \]

\[ L_2 = \frac{sL_2 C}{sL_2} \]

\[ L_3 = \frac{R_2}{sL_2} \]

Two nontouching loops: \[ L_1 L_3 = \frac{R_2}{s^2 L_2 C(R_1 + sL_1)} \]

\[ \Delta = 1 - [L_1 + L_2 + L_3] + L_1 L_3 \]

Example 12.31: Draw the signal flow graph of series RLC circuit shown in fig. 1.30 and find the transfer function.

Solution: Step 1: Draw the transform network

\[ E_i(s) \longrightarrow 1/sC \longrightarrow E_i(s) \]

\[ \frac{R}{sL} \longrightarrow \frac{1}{sC} \longrightarrow \frac{1}{sL} \]

Step 2: Write the equation

\[ i(s) = \frac{E_i(s) - E_i(s)}{R + sL} = \frac{E_i(s)}{R + sL} - \frac{E_i(s)}{R + sL} \]

\[ E_i(s) = i(s) \cdot \frac{1}{sC} \]

Step 3: For each equations obtained in step 2, draw SFG. Select the node variables as \( i(s) \), \( E_i(s) \) and \( E_i(s) \).

\[ E_i(s) \quad \frac{1}{R + sL} \quad i(s) \quad E_i(s) \]

Step 4: Combine the SFG. Obtained in step 3

\[ E_i(s) \quad \frac{1}{R + sL} \quad i(s) \quad \frac{1}{sC} \quad E_i(s) \]
Step 5: Forward path

\[ \frac{1}{1 + \frac{1}{R + sL}} \frac{1}{sC} \Delta_1 = 1 \]

Individual loop \( L_1 = \frac{1}{(R + sL)(sC)} \)

Non touching loops: None

\[ \Delta = 1 - L_1 = 1 + \frac{1}{(R + sL)(sC)} + \frac{(R + sL)(sC) + 1}{(R + sL)(sC)} \]

\[ \frac{C(s)}{R(s)} = \frac{\frac{sC(R + sL)}{1 + (sC)(R + sL)}}{\frac{sC(R + sL)}{sC(R + sL)}} = \frac{1}{1 + RC + s^2L} \]

\[ \frac{C(s)}{R(s)} = \frac{1}{s^2LC + sRC + 1} \text{ Ans.} \]

Example 12.32. Draw the SFG of lag network shown in Fig. 6.4 and find the transfer function.

Solution: Step 1: Draw the transform network

Step 2: Write the equations

\[ I(s) = \frac{E_1(s)}{R_1} \frac{E_2(s)}{R_2} \]

\[ E_1(s) = I(s) \left[ \frac{R_2}{sC} + 1 \right] \]

Step 3: Draw SFG for each equation

\[ \frac{E_1(s)}{I(s)} \]

\[ \frac{E_2(s)}{sR_2C + 1} \]

Step 4: Combine SFG obtained in step 3

Example 12.33. For the given circuit, draw SFG and find out the transfer function.

Solution: Draw the transform network (Fig. 12.60)

\[ I_4(s) = \frac{V_1(s)}{sC_1} \frac{V_2(s)}{sC_1} \]

\[ V_1(s) = sC_1 V_1(s) - sC_1 V_4(s) \]

\[ V_2(s) = [I_1(s) - I_2(s)] R_1 \]

\[ I_1(s) = I_1(s) R_1 - I_2(s) R_1 \]

\[ I_2(s) = sC_2 V_1(s) - sC_1 V_4(s) \]

\[ V_4(s) = I_2(s) R_2 \]
Example 12.34. For the system represented by the following equations, find the transfer function by SFG.

\[ x = x + \beta_1 u \]
\[ x = x_1 + x_2 + \beta_2 u \]
\[ z = x_1 + x_2 + \beta_3 u \]

Solution: Let
\[ u = input \]
Take laplace transform of above equations

\[ X(s) = X_1(s) + \beta_3 u(s) \]
\[ sX_1(s) = -a_1 X_1(s) + X_2(s) + \beta_2 u(s) \]
\[ sX_2(s) = -a_2 X_1(s) + \beta_1 u(s) \]

**L_1:**

\[ L_1 = -\frac{a_2}{s + a_1} \]

\[ \Delta = 1 - (L_1) = 1 + \frac{a_2}{s + a_1} \]

\[ E(s) = Z(s) \frac{I(s)}{s} \]

\[ E(s) = -Z_I(s) \]

\[ C(s) = \frac{\beta_1 s + \beta_2 s + \beta_3}{s(s + a_1)(s + a_2)} \]

\[ C(s) = \frac{\beta_1}{s(s + a_1)} + \frac{\beta_2}{s(s + a_2)} + \frac{\beta_3}{s + a_1} \]

\[ \text{Ans.} \]

**Example 12.36.** For the given op-amp circuit find the transfer function \[ \frac{E_o(s)}{E_i(s)} \]

**Solution:** Transform network of the given circuit (Fig. 12.63).

\[ Z_1(s) = R_1 \]

\[ Z_2(s) = \frac{R_2}{R_2 + \frac{1}{sC}} = \frac{R_2}{1 + sR_2C} \]

\[ \frac{E_o(s)}{E_i(s)} = \frac{Z_2(s)}{Z_1(s)} \]

\[ \frac{E_o(s)}{E_i(s)} = -\frac{R_2}{R_1(1 + sR_2C)} \]

This is the required transfer function.
Example 12.37. Find the transfer function of inverting amplifier as shown in fig.

Solution:

\[ i_1 = \frac{V_i - V_o}{R_1} \quad \text{and} \quad i_2 = \frac{V_o - V_i}{R_2} \]

Laplace transform:

\[ \frac{E_i(s)}{R_1} = -\frac{E_o(s)}{R_2} \]

\[ \frac{E_o(s)}{E_i(s)} = \frac{R_2}{R_1} \text{ Required transfer function} \]

Example 12.38. Determine the transfer function of given circuit using op-amp as shown in Fig. 12.65.

Solution:

\[ i_1(t) = \frac{V_i(t) - V_o(t)}{R} \]

\[ i_2(t) = \frac{C}{dt} (V_i(t) - V_o(t)) \]

Laplace transform of above equations:

\[ i_1(s) = \frac{1}{R} [V_i(s) - V_o(s)] \]

\[ i_2(s) = sC[V_i(s) - V_o(s)] \]

Since:

\[ V_i(s) = 0 \quad \Rightarrow \quad i_1(s) = i_2(s) \]

\[ i_1(s) = \frac{V_i(s)}{R} \]

\[ V_o(s) = \frac{R_2}{R_2 + \frac{1}{sC}} E_i(s) \]

\[ V_A(s) = V_p(s) \]

\[ I_1(s) = \frac{E_i(s) - V_A(s)}{R_1} \]

\[ I_2(s) = \frac{V_A(s) - E_o(s)}{R_1} \]

Since

\[ \frac{E_1(s) - V_A(s)}{R_1} = \frac{V_A(s) - E_o(s)}{R_1} \]

\[ 2V_A(s) = E_i(s) + E_o(s) \]
Example 12.40. Find the transfer function of the circuit shown in Fig. 12.67.

Solution: Transform the network

\[
Z_1 = \frac{R_2}{1 + sR_2C_1}
\]

\[
Z_2 = R_2
\]

\[
V_A = 0
\]

\[
I_1(s) = \frac{E(s) - V_A(s)}{Z_1} = \frac{1 + sR_2C_1}{R_1} E(s)
\]

\[
I_2(s) = -\frac{E(s)}{R_2}
\]

since, \( I_1(s) = I_2(s) \)

\[
E(s) = -\frac{R_2(1 + sR_2C_1)}{R_1} E(s)
\]

or,

\[
\frac{E(s)}{E(s)} = -\frac{R_2(1 + sR_2C_1)}{R_1}
\]

since,

\[
I_1(s) = \frac{V(s)}{R_1}
\]

\[
I_2(s) = \frac{V(s)}{1/sC_1} = -sC_1 V(s)
\]

\[
I_1(s) = I_2(s)
\]
Example 12.42. Find the transfer function of the given circuit shown in Fig. 12.69.

Solution: Transform the network shown in Fig. 12.69 a

\[ Z_1(s) = \frac{R_l}{\frac{1}{C_1} \left( R_1 + \frac{1}{sC_1} \right)} = \frac{R_l}{1 + sR_1C_1} \]

This is the required transfer function.

Example 12.43. For the given signal flow graph of a system find the expressions for the outputs \( E(s) \) and \( F(s) \). Also determine the condition that makes \( C_1 \) independent of \( R_2 \) and \( C_2 \).
Expression for $C_1$:

$g_1 = R_1 G_2$
$g_2 = R_2 G_4 H_3$
$g_3 = R_3 G_3$
$g_4 = R_4 G_1 H_1$

$\Delta_1 = 1 - G_1 H_3$
$\Delta_2 = G_2 H_2$
$\Delta_3 = G_3 H_1$
$\Delta_4 = 1 - G_4 H_1$

Individual loops:

$L_1 = G_1 H_3$
$L_2 = G_2 H_2$
$L_3 = G_3 H_1$
$L_4 = G_4 H_1$
$L_5 = G_5 H_3 C_3 H_4$
$L_6 = G_6 H_1 H_2$

Two non-touching loops

$L_1 L_4 = G_1 H_3 G_2 H_4$
$L_2 L_3 = G_2 H_2 G_3 H_1$

$\Delta = 1 - (L_1 + L_2 + L_3 + L_4 + L_5 + L_6) + (L_4 L_4 + L_2 L_3)$

$C_1 =$

$\frac{g_1 \Delta_1 + g_2 \Delta_2 + g_3 \Delta_3 + g_4 \Delta_4}{\Delta}$

$C_2 =$

$\frac{R_2 G_2 (1 - G_2 H_2) + R_4 G_4 G_2 H_2 + R_2 G_2 (1 - G_2 H_2) + R_2 G_2 H_2}{1 - G_2 H_2 + G_2 H_2 + G_3 H_1 + G_4 H_1 + G_5 H_3 C_3 H_4 + G_5 H_2 H_1 - G_5 C_3 H_4 H_2 - G_5 G_2 H_2 H_2}$

Condition $C_1$ Independent of $R_2$

From the expression of $C_1$

$R_2 G_2 (1 - G_2 H_2) + 2 G_2 C_2 H_2 = 0$

$R_2 G_2 (1 - G_2 H_2) + G_3 G_2 H_2 = 0$

$G_2 G_2 H_2 + G_2 G_2 H_2 = 0$

$G_2 = (-G_2 G_4 + C_2 G_3) H_2$

$H_2 = \frac{-G_2}{G_2 G_2 - G_2 G_4}$

Similarly $C_2$ independent of $R_1$ when

$H_1 = \frac{-G_3}{G_3 G_2 - G_3 G_4}$

Example 12.44. For the given circuit obtain the state equation. Take voltage across the capacitors $V_1(t)$ & $V_2(t)$ as two state variables and the current through the resistor as output variable.
Example 12.46. Find the state transition matrix $\Psi(t)$ and the characteristic equation of $A$.

$$A = \begin{bmatrix} -3 & 0 \\ 0 & -3 \end{bmatrix}$$

Solution: The characteristic equation is defined as

$$|sI - A| = 0$$

$$sI - A = s \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} -3 & 0 \\ 0 & -3 \end{bmatrix} = \begin{bmatrix} s + 3 & 0 \\ 0 & s + 3 \end{bmatrix}$$

$$|sI - A| = \det(s + 3)(s + 3) = s^2 + 6s + 9 = 0$$

:. Required characteristic equation $\Rightarrow s^2 + 6s + 9 = 0$  

Ans.

Calculation of $\Psi(t)$:

$$[sI - A]^{-1} = \frac{1}{s + 3} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\Psi(t) = e^{sI} [sI - A]^{-1} = \begin{bmatrix} e^{s + 3} & 0 \\ 0 & e^{s + 3} \end{bmatrix}$$

$$\Psi(t) = \begin{bmatrix} e^{-3t} & 0 \\ 0 & e^{-3t} \end{bmatrix}$$  

Ans.
Solution : We know that
\[ G(s) = C(sI - A)^{-1} B \]
\[ A = \begin{bmatrix} -1 & 1 \\ 0 & -2 \end{bmatrix} \]
\[ sI - A = \begin{bmatrix} s+1 & 1 \\ 0 & s+2 \end{bmatrix} \]
\[ [sI - A]^{-1} = \frac{1}{s^2 + 3s + 2} \begin{bmatrix} s+2 & 1 \\ 0 & s+1 \end{bmatrix} \]
\[ C[sI-A]^{-1}B = \frac{1}{s^2 + 3s + 2} \begin{bmatrix} s+2 & 2s+3 \\ s+2 & 1 \\ s+2 & s+2 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} \]
\[ G(s) = \frac{s^3 - 3s^2 - 9s + 2}{s^2 + 3s + 2} \]

This is the required transfer function

Ans.

Example 12.49. The mechanical system shown in fig 12.5 choose \( x_1 \) and \( x_2 \) of mass \( M_1 \) and \( M_2 \) as state variables as well as output obtain the state model.

Solution : The system equations are
\[ F = M_1 \frac{d^2 x_1}{dt^2} + B_1 \frac{dx_1}{dt} + B_2 \frac{dx_2}{dt} \]
\[ 0 = M_2 \frac{d^2 x_2}{dt^2} + K x_2 + B_2 \frac{dx_1}{dt} - B_2 x_2 \]

Select the state variable as
\[ x_1 = Z_1 \]
\[ x_2 = Z_3 \]
\[ x_1 = Z_2 \]
\[ x_2 = Z_3 + Z_4 \]
\[ x_1 = \ddot{Z}_2 \]
\[ x_2 = \ddot{Z}_4 \]
equation (3) and (4) can be written as
\[ F = M_1 \ddot{Z}_2 + (B_1 + B_2) Z_2 - B_2 Z_4 \]

The state equations are
\[ \dot{Z}_1 = Z_2 \]
\[ \ddot{Z}_2 = \frac{F}{M_1} \left( \frac{B_1 + B_2}{M_1} \right) Z_2 - \frac{B_2}{M_1} Z_4 \]
\[ \ddot{Z}_3 = Z_4 \]
\[ \ddot{Z}_4 = \frac{K}{M_2} Z_2 - \frac{B_2}{M_2} Z_4 - B_2 Z_2 \]

Select displacement of mass \( M_1 \) and \( M_2 \) as output variables
\[ y_1 = x_1 = Z_1 \]
\[ y_2 = x_2 = Z_3 \]

Example 12.50. Find the characteristic equation when
\[ A = \begin{bmatrix} -5 & 1 & 0 \\ 0 & -5 & 1 \\ 0 & 0 & -5 \end{bmatrix} \]

Solution : The characteristic equation \( \Rightarrow |sI - A| = 0 \)
\[ s^3 + 15s^2 + 75s + 125 = 0 \]

Required characteristic equation
\[ s^3 + 15s^2 + 75s + 125 = 0 \] Ans.
Example 12.51. Write short note characteristic equation for differential equation, transfer function and for state equation.

Solution: Characteristic Equation from Differential Equation

Consider the differential equation of linear time-invariant system

$$\frac{d^n x(t)}{dt^n} + a_{n-1} \frac{d^{n-1} x(t)}{dt^{n-1}} + \cdots + a_0 x(t) = b_m \frac{d^m u(t)}{dt^m} + \cdots + b_0 u(t)$$

Laplace transform of above equation

$$(s^n + a_{n-1} s^{n-1} + \cdots + a_0) X(s) = (b_m s^m + \cdots + b_0) U(s)$$

The characteristic equation can be obtained by setting the homogeneous part to zero.

So, the characteristic equation will be

$$s^n + a_{n-1} s^{n-1} + \cdots + a_0 = 0$$

For example:

The characteristic equation of

$$\frac{d^4 x(t)}{dt^4} + 3 \frac{d^3 x(t)}{dt^3} + 2 \frac{d^2 x(t)}{dt^2} + 5 \frac{d x(t)}{dt} + 5 x(t) = u(t)$$

will be

$$s^4 + 3s^3 + 2s^2 + 5s + 5 = 0$$

Characteristic Equation from Transfer Function

The characteristic equation from the transfer function can be obtained by equating the denominator polynomial to zero.

For example:

$$\frac{C(s)}{R(s)} = G(s) = \frac{b_m s^m + b_{m-1} s^{m-1} + \cdots + b_0}{a_n s^n + \cdots + a_0}$$

The characteristic equation is

$$a_n s^n + \cdots + a_0 = 0$$

If

$$G(s) = \frac{K}{s(1.4s)(1+0.25s)}$$

$$H(s) = 1$$

Then

$$\frac{C(s)}{R(s)} = \frac{G(s)}{1 + G(s) H(s)}$$

The characteristic equation

$$1 + \frac{K}{s(1+0.4s)(1+0.25s)} = 0$$

$$s (1+0.4s)(1+0.25s) + K = 0$$

is the required characteristic equation.

Character Equation from State Equation:

From equation 8.38

$$G(s) = C [sI - A]^{-1} B + D = C \frac{\text{adj} (sI - A) B + D}{[sI - A]}$$

$$G(s) = \frac{C \text{adj} (sI - A) B + [sI - A] D}{[sI - A]}$$

Example 12.52. For the circuit shown in fig. 1.114, if $R_1 = R_2 = R$ and $C_1 = C_2 = C$, show that the system is always overdamped.

Solution: Transfer function of the given circuit

$$\frac{C(s)}{R(s)} = \frac{1}{s^2 + \frac{3}{RC} s + \frac{1}{C}}$$

Compare with

$$\frac{C(s)}{R(s)} = \frac{\frac{\omega_n^2}{s^2 + 2\zeta \omega_n s + \omega_n^2}}$$

$$\omega_n = \frac{1}{\sqrt{RC}}$$

$$2\zeta \omega_n = \frac{3}{RC}$$

Since, $\zeta$ is greater than one, hence the system is always overdamped.

Example 12.53. The differential equation of a control system is

$$\frac{d^3 Y(t)}{dt^3} + 9 \frac{d^2 Y(t)}{dt^2} + 20 \frac{d Y(t)}{dt} + 20 Y(t) = 20 X(t)$$

Solve the output response for unit step input $X(t)$ and $Y(t)$ are input and output respectively.

Solution: Given differential equation is

$$\frac{d^3 Y(t)}{dt^3} + 9 \frac{d^2 Y(t)}{dt^2} + 20 \frac{d Y(t)}{dt} + 20 Y(t) = 20 X(t)$$

Replace transform of above equation

$$s^3 Y(s) + 9 s^2 Y(s) + 20 s Y(s) + 20 Y(s) = 20 X(s)$$

$$\frac{Y(s)}{X(s)} = \frac{20}{s^3 + 9s^2 + 20}$$
Example 12.54. The output of a control system is
\[ C(t) = 1 + 0.25 e^{-50t} - 1.25 e^{-10t} \]

(i) Obtain the expression for closed loop transfer function of the system.

(ii) Determine undamped natural frequency & damping ratio. Assume unit step input.

Solution:
\[ C(s) = \frac{1}{s} + \frac{0.25}{s+50} + \frac{1.25}{s+10} = \frac{500}{(s+10)(s+50)} \]

since,
\[ R(s) = \frac{1}{s} \text{ given} \]

\[ \therefore \frac{C(s)}{R(s)} = \frac{500}{(s+10)(s+50)} \text{ Ans.} \]

:. closed loop transfer function
\[ \frac{C(s)}{R(s)} = \frac{500}{s^2 + 60s + 500} \]

.. compare with \( \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2} \)

.. \( \omega_n^2 = 500 \)

.. \( \omega_n = 22.36 \text{ rad/sec.} \)

.. \( 2\zeta\omega_n = 60 \)

.. \( 2\zeta = 22.36 \)

.. \( \zeta = 1.34 \text{ Ans.} \)

Example 12.55. Find impulse response of the given circuit.

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Fig. 12.73.
Example 12.56. Obtain the step response of series RLC circuit with \( R = 1 \Omega, L = 1 \, \text{H} \) and \( C = \frac{1}{2} \text{F} \), output taken across 'R'. Assume input as a step of 10v.

Solution:

The transform network is shown in fig.

Apply KVL

\[
E_i(s) = \left( R + sL + \frac{1}{sC} \right) i(s)
\]

\[
E_i(s) = RI(s)
\]

\[
E_i(s) = \frac{RCs}{s^2LC + sRC + 1}
\]

\[
E_i(s) = \frac{10}{s} \quad \text{(given)} \quad \& \quad R = 1, \quad L = 1, \quad C = 1
\]

\[
E_i(s) = \frac{10}{s^2 + s + 1} = \frac{10}{\left( s + \frac{1}{2} + j\frac{\sqrt{3}}{2} \right) \left( s + \frac{1}{2} - j\frac{\sqrt{3}}{2} \right)}
\]

\[
E_i(s) = \frac{A}{\left( s + \frac{1}{2} + j\frac{\sqrt{3}}{2} \right)} + \frac{B}{\left( s + \frac{1}{2} - j\frac{\sqrt{3}}{2} \right)}
\]

\[
E_i(s) = \frac{10}{\left( s + \frac{1}{2} + j\frac{\sqrt{3}}{2} \right) \left( s + \frac{1}{2} - j\frac{\sqrt{3}}{2} \right)}
\]

\[
E_i(s) = \frac{-0.0386 x \left( s + \frac{1}{2} + j\frac{\sqrt{3}}{2} \right)}{\left( s + \frac{1}{2} - j\frac{\sqrt{3}}{2} \right)}
\]

\[
E_i(s) = 0.5386 \frac{e^{-1.866s}}{s + 1.866} - 0.0386 e^{-0.1339s}
\]  

Ans.

Sample 12.57. For a control system shown in fig 12.75. Find the values of \( K \) and \( K_i \) so that the rising ratio (\( t_r \)) of system is 0.6 and setting time (\( t_s \)) is 0.1sec. Use \( t_i = 3.2/C_{0v} \). Assume unit step input.

\[
\frac{R(s)}{R(s)} = \frac{100}{s(1 + 0.2s)}
\]

\[
\frac{R(s)}{R(s)} = \frac{100}{1 + 0.2s + 100K_i}
\]

\[
\frac{R(s)}{R(s)} = \frac{100}{1 + 0.2s + 100K_i}
\]

Solution: First solve inner closed loop.
Now three blocks are in series

\[ C(s) = \frac{25K}{R(s)} = \frac{25K}{s^2 + 5s + 100K} \]

\[ \omega_n^2 = 25k \]
\[ 2\zeta \omega_n = 5(1 + 100K) \]

Given
\[ t = \frac{3.2}{\zeta \omega_n} \]
\[ 0.1 = \frac{3.2}{0.6 \times 5\sqrt{K}} \]
\[ K = 113.78 \]
\[ \omega_n = 5\sqrt{K} = 5\sqrt{113.78} = 53.33 \text{ rad/sec} \]
\[ \omega_n = 53.33 \text{ rad/s} \text{ Ans.} \]

Example 12.58. The characteristic equation of a system is given by \( s^6 + 3s^5 + 4s^4 + 5s^3 + 2s^2 + 35 + 2 = 0 \). Comment on stability.

Solution:

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<th>4</th>
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<td>5</td>
<td>2</td>
</tr>
<tr>
<td>( S^5 )</td>
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<td>( S^1 )</td>
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<td>0</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>

One row becomes zero, the auxiliary eq. \( A(s) \)
\[ A(s) = 2s^4 + 1 = 0 \]

or, \( S^4 + 2S^2 + 1 = 0 \)

\[ \frac{dA(s)}{ds} = 4S^3 + 4S \]

<table>
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<tr>
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<th>4</th>
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<tr>
<td>( S^0 )</td>
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</tr>
</tbody>
</table>

Again one row becomes zero, the auxiliary equation \( B(s) \)
\[ B(s) = 2s^2 + 2 = 0 \]

\[ \frac{dB(s)}{ds} = 4s = 0 \]

All elements in first column are of the same sign, therefore no root is in right half of s-plane, but there are two row of zeros, so check the system for marginally stable or unstable.

From first auxiliary equation
\[ A(s) = s^4 + 2s^2 + 1 \]

Put \( s^2 = x \)
\[ x^2 + 2x + 1 = 0 \]
\[ (x + 1)^2 = 0 \]
\[ x = -1 \]

\[ s^2 = -1 \]
\[ S = \pm j \]
\[ S_1 = +j \]
\[ S_2 = -j \]
\[ S_3 = +j \]
\[ S_4 = -j \]

From second auxiliary equation
\[ 2s^2 + 2 = 0 \]
\[ s^2 = -1 \]
\[ S = \pm j \]
\[ S_5 = +j \& S_6 = -j \]

On imaginary axis the roots are repeated, hence system is unstable.

Example 12.59. Determine the stability of a system with characteristic equation.
\[ s^6 + 4s^4 + 2s^3 + 8s^2 + s + 4 = 0 \]

Solution:

<table>
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<tr>
<th>Row</th>
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<th>3</th>
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</tr>
<tr>
<td>( s^3 )</td>
<td>-</td>
<td>-</td>
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</table>

One row becomes zero, use auxiliary equation
\[ A(s) = 4s^4 + 8s^2 + 4 = 0 \]

or, \( s^4 + 8s^2 + 1 = 0 \)

Put \( S^2 = x \)
\[ x^2 + 2x + 1 = 0 \]
\[ (x + 1)^2 = 0 \]
\[ x = -1 \]

\[ S = \pm j \]
\[ S_1 = +j \]
\[ S_2 = -j \]

\[ S_3 = +j \]
\[ S_4 = -j \]

\[ S_5 = +j \& S_6 = -j \]
**Example 12.60.** Determine stability of a system having characteristic equation

\[ s^6 + s^5 + 10s^4 + 24 s^3 + 20 s^2 + 15s + 10 = 0 \]

**Solution:**

\[
\begin{align*}
S^6 & = 0 \\
S^5 & = 10 \\
S^4 & = 15 \\
S^3 & = 10 \\
S^2 & = 1 \\
S^1 & = 0 \\
S^0 & = 10
\end{align*}
\]

No. of sign changes = 2

Hence, system is unstable.

**Example 12.61.** Determine number of roots with positive real parts, zero real part and negative real part for the following polynomial equation using Routh criterion.

\[ \phi(s) = s^4 + 2s^3 + 8s^2 + 10s + 15 \]

**Solution:**

\[
\begin{align*}
S^4 & = 1 \\
S^3 & = 2 \\
S^2 & = 3 \\
S^1 & = 0 \\
S^0 & = 15
\end{align*}
\]

Auxiliary equation

\[ A(s) = 3s^2 + 15 \]

\[ \frac{dA(s)}{ds} = 6s \]

\[
\begin{align*}
S^4 & = 1 \\
S^3 & = 2 \\
S^2 & = 3 \\
S^1 & = 6 \\
S^0 & = 15
\end{align*}
\]

Multiply the characteristic equation by \((s + 1)\)

\[
(6 + 1)(2s^3 + 7s^2 + 9s + 4s^2 + 2s + 1) = 0
\]

No. of roots with positive real part = 2

No. of roots with negative real part = 2

No. of roots with zero real part = 2

No. of sign changes = 2

Example 12.62. Determine the number of roots with positive real part, no. of roots with zero real part and the number of roots with negative real part for the following equations, apply Routh-Hurwitz criterion.

(a) \[ 2s^5 + s^4 + 6s^3 + 3s^2 + s + 1 = 0 \]

(b) \[ s^5 - s^4 + 3s^3 - 2s^2 - 8s + 8 = 0 \]

**Solution:**

(a) \[ 2s^3 + s^2 + 6s^3 + 3s^2 + s + 1 = 0 \]

(b) \[ 2s^3 + 7s^2 + 9s + 4s^2 + 2s + 1 = 0 \]

No. of roots with positive real part = 2

No. of roots with negative real part = 2

No. of roots with zero real part = 2

No. of sign changes = 2
Example 12.63. Apply Routh-Hurwitz criterion to determine (1) number of roots with positive real part (2) the number of roots with zero real parts (3) the number of roots with negative real parts for the following equation.
(a) \( s^4 + 2s^3 + 7s^2 + 10s + 10 = 0 \)

Solution:
\[
\begin{align*}
S^4 & = 1 & 7 & 10 & 0 \\
S^3 & = 2 & 10 \\
S^2 & = 2 & 10 \\
S^1 & = 0 \\
S^0 & = 0
\end{align*}
\]
Auxiliary equation \( A(s) = 2s^2 + 10 = 0 \)

Example 12.64. Find the number of roots with positive real part, zero real part and negative real parts of the following characteristic equation
\( s^3 + 5s^2 + 2s + 10 = 0 \)

Solution:
\[
\begin{align*}
\frac{dA(s)}{ds} &= 4s \\
S^3 &= 1 & 10 \\
S^2 &= 2 & 10 \\
S^1 &= 2 & 10 \\
S^0 &= 10
\end{align*}
\]

No sign change in first column, system may be stable. Solve the auxiliary equation
\[ 2s^2 + 10 = 0 \]
\[ s = \pm j\sqrt{5} \]

Hence, roots are non-repeated, the system is marginally stable. The given equation can be written in factored form as
\[ (s + 5)(s^2 + 2s + 2) = 0 \]

No. of roots with positive real part = 0
No. of roots with zero real part = 2
No. of roots with negative real part = 2

Example 12.65. Find the number of roots with positive real part, zero real part and negative real parts of the following characteristic equation
\[ (s^2 + 5s + 10)(s + 1) = 0 \]

Solution:
\[
\begin{align*}
S^2 + 5S + 10 & = 0 \\
S^1 & = 1 & 0 & 10 \\
S^0 & = 1 & 2 & 10 \\
S^0 & = 127 \\
S^0 & = 10
\end{align*}
\]

No. of sign change = 2
No. of roots with positive real part = No. of sign change = 2

The given characteristic equation in factored form can be written as:
\[
\begin{align*}
& (s + 5)(s^2 + 2) \\
& (s + 5)(s + 1.26)(s^2 - 1.26s + 1.59) \\
& (s + 5)(s + 1.26)(s - 0.63 - j1.09)(s - 0.63 + j1.09) = 0
\end{align*}
\]
Example 12.65. Determine the values of K for open loop transfer function

\[ G(s) H(s) = \frac{K}{s(1+0.1s)(1+s)} \] so that

(a) the gain margin is 15 db and

(b) Phase margin is 60°.

Solution : Step 1:

\[ G(s) H(s) = \frac{K}{s(1+0.1s)(1+s)} \]

corner frequencies are

\[ \omega_1 = \frac{1}{0.1} = 10 \text{ rad/sec.} \quad \text{and} \quad \omega_2 = \frac{1}{1} = 1 \text{ rad/sec.} \]

Step 2: Draw the magnitude plot without K. Since there is a pole at origin, the initial slope of the magnitude plot will be -20 db/dec.

(i) Draw a line having slope -20 db/dec. (parallel to the I slope marker on top of the corner) passing through \( \omega = 1 \) upto first corner frequency 1 rad/sec.

(ii) At \( \omega = 1 \), a simple pole occurs, the resultant slope of next line will be -20 + (-20) = -40 db/dec. Draw a line having slope -40 db/dec. (parallel to II slope marker) upto II corner frequency 10 rad/sec.

(iii) At \( \omega = 10 \), a simple pole occurs, the resultant slope of next line will be -40 + (-20) = -60 db/dec. Draw a line having slope -60 db/dec. from second corner frequency towards infinity as there is no pole or zero.

Step 3 : Draw phase plot

\[ \phi = -90^\circ - \tan^{-1} 0.1 \omega - \tan^{-1} \omega \]

<table>
<thead>
<tr>
<th>( \omega )</th>
<th>( \text{Arg}(j\omega) )</th>
<th>( \text{Arg}(1+j0.1\omega) )</th>
<th>( \text{Arg}(1+j\omega) )</th>
<th>Resultant ( \phi = \phi_1 + \phi_2 + \phi_3 )</th>
</tr>
</thead>
<tbody>
<tr>
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<td>-90°</td>
<td>-1.14°</td>
<td>-11.3°</td>
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<td>-90°</td>
<td>-4.6°</td>
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<tr>
<td>1.0</td>
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<td>-5.7°</td>
<td>-45°</td>
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<tr>
<td>3.0</td>
<td>-90°</td>
<td>-16.69°</td>
<td>-71.56°</td>
<td>-178.25°</td>
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<tr>
<td>4.0</td>
<td>-90°</td>
<td>-21.8°</td>
<td>-75.96°</td>
<td>-187.76°</td>
</tr>
</tbody>
</table>

Step 4 : From Bode plot (plot 1), gain margin available = 21 db

Required gain margin = 15 db.

For required gain margin, shift the magnitude plot upwards by 21 - 15 = 6 db. 30, shift point \( s = 6 \) db upward shift taken as positive and downward shift negative.

\[ P.M. = 180^\circ + \arg(G(j\omega) H(j\omega)) \]

\[ + 60^\circ = 180^\circ + \arg(G(j\omega) H(j\omega)) \]

\[ \arg(G(j\omega) H(j\omega)) = 60^\circ - 180^\circ = -120^\circ \]

Find this point on phase plot i.e., locate D at -120°. Extend a line from D on magnitude plot to intersect at E, but this point should be at F for P.M. (+60°). So shift the magnitude plot downwards.

\[ \frac{EF}{E} = \frac{6}{20 \log_{10} K} \]

\[ K = 0.501 \]

Example 12.66. Determine the value of K for open loop transfer function

\[ G(j\omega) H(j\omega) = \frac{K}{j\omega (1+0.1\omega)(1+j0.05\omega)} \]

such that

(i) the gain margin is 20 db

(ii) the phase margin is 30°.

Solution : Corner frequencies are

\[ \omega_1 = \frac{1}{0.1} = 10 \text{ rad/sec.} \quad \text{and} \quad \omega_2 = \frac{1}{0.05} = 20 \text{ rad/sec.} \]

Step 1: Draw the magnitude plot without K. The procedure is same as in example 12.65.

Step 2 : Draw the phase plot

\[ \phi = -90^\circ - \tan^{-1} 0.1 \omega - \tan^{-1} \omega \]

<table>
<thead>
<tr>
<th>( \omega )</th>
<th>( \text{Arg}(j\omega) )</th>
<th>( \text{Arg}(1+j0.1\omega) )</th>
<th>( \text{Arg}(1+j0.05\omega) )</th>
<th>Resultant ( \phi = \phi_1 + \phi_2 + \phi_3 )</th>
</tr>
</thead>
<tbody>
<tr>
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<td>-21.8°</td>
<td>-11.3°</td>
<td>-123.1°</td>
</tr>
<tr>
<td>6.0</td>
<td>-90°</td>
<td>-30.96°</td>
<td>-16.69°</td>
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<tr>
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<td>-54.46°</td>
<td>-35°</td>
<td>-179.46°</td>
</tr>
<tr>
<td>18.0</td>
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<td>-60.9°</td>
<td>-42°</td>
<td>-192.9</td>
</tr>
<tr>
<td>20.0</td>
<td>-90°</td>
<td>-63.43°</td>
<td>-45°</td>
<td>-198.43</td>
</tr>
</tbody>
</table>
Step 1: From Bode plot
Gain margin available = 27 db
Gain margin required = 20 db
For required gain margin, shift the magnitude plot upwards i.e., shift the point A to B = +7 db

\[ 7 \text{ db} = 20 \log_{10} K \]
\[ K = 2.23 \]

Step 2:

\[ \text{P.M.} = 180^\circ + \frac{\sqrt{G(j\omega)H(j\omega)}}{1 + \sqrt{G(j\omega)H(j\omega)}} \]
\[ 30^\circ = 180^\circ + \frac{\sqrt{G(j\omega)H(j\omega)}}{1 + \sqrt{G(j\omega)H(j\omega)}} \]
\[ \sqrt{G(j\omega)H(j\omega)} = 30 - 180 = -150^\circ \]

Find this point on phase plot i.e., locate D at -150°. Extend the line from D to the magnitude plot which intersects the magnitude plot at E. For required phase margin shift the magnitude plot upwards. For P.M. = 30° the magnitude plot should intersect 0 db line at F. So, the total shift is EF.

\[ EF = 19 \text{ db} \]
\[ 20 \log K = 19 \]
\[ K = 8.91 \]

Result:

(i) Value of K for gain margin 20 db = 2.23
(ii) Value of K for phase margin 30° = 8.91

Example 12.67. Determine the marginal value of K analytically for unity feedback system having

\[ G(s)H(s) = \frac{K}{s(s + 5)(s + 10)} \]
so that the system will marginally stable.

Solution: For marginally stable system
Phase margin = Gain margin = 0
Phase cross-over frequency = Gain crossover frequency.

We know that

\[ \text{P.M.} = 180^\circ + \frac{\sqrt{G(j\omega)H(j\omega)}}{1 + \sqrt{G(j\omega)H(j\omega)}} \]
\[ \text{P.M.} = 0^\circ \]
\[ 0^\circ = 180^\circ + \frac{\sqrt{G(j\omega)H(j\omega)}}{1 + \sqrt{G(j\omega)H(j\omega)}} \]
\[ \frac{\sqrt{G(j\omega)H(j\omega)}}{1 + \sqrt{G(j\omega)H(j\omega)}} = -180^\circ \]
\[ -90^\circ - \tan^{-1} 0.2\omega - \tan^{-1} 0.1\omega = -180^\circ \]
\[ \tan^{-1} 0.2\omega + \tan^{-1} 0.1\omega = 90^\circ \]

Taking tan on both sides
\[ \tan (\tan^{-1} 0.2\omega + \tan^{-1} 0.1\omega) = \tan 90^\circ \]
\[ \frac{0.2\omega + 0.1\omega}{1 - (0.2\omega)(0.1\omega)} = \infty \]
\[ 1 - 0.02 \omega^2 = 0 \]
\[ \omega = 7.07 \text{ rad/sec.} \]

This is the gain crossover frequency \( \omega_c = 7.07 \text{ rad/sec.} \) At gain crossover frequency

\[ \frac{G(j\omega)H(j\omega)}{K} = 1 \]

\[ \frac{K}{\sqrt{7.07}[5 + j7.07][10 + j7.07]} = 1 \]

\[ \frac{K}{7.07 \times 8.66 \times 12.24} = 1 \]

\[ K = 749.82 \]

\[ G.M. = 20 \log_{10} \left| \frac{1}{G(j\omega)H(j\omega)} \right|_{\omega = \omega_c} \]

Put \( G(j\omega)H(j\omega) = x \)

\[ G.M. = 20 \log_{10} \frac{1}{x} \]

\[ 0 = 20 \log_{10} \frac{1}{x} \]

\[ \frac{1}{x} = 1 \quad \text{or,} \quad x = 1 \]

\[ G(j\omega)H(j\omega) = \frac{K}{(j\omega)(5 + j\omega)(10 + j\omega)} \]

Rationalize above function to find \( x \), at point \( x \quad G(j\omega)H(j\omega) = -180^\circ \), i.e., imaginary part is

\[ G(j\omega)H(j\omega) = \frac{K(-j\omega)(5 - j\omega)(10 - j\omega)}{(j\omega)(5 + j\omega)(10 + j\omega)(5 - j\omega)(10 - j\omega)} \]

\[ \frac{15K}{(25 + \omega^2)(100 + \omega^2)} - j \frac{K\omega(50 - \omega^2)}{\omega^2(25 + \omega^2)(100 + \omega^2)} \]

Equate the imaginary part to zero

\[ 50 - \omega^2 = 0 \]

or,

\[ \omega^2 = 50 \]

\[ \omega = 7.07 \text{ rad/sec.} \]

This is the phase crossover frequency \( \omega_p = 7.07 \text{ rad/sec.} \) Put the value of \( \omega_p \) in real part, the point \( x \) is

\[ x = \frac{15K}{(25 + 50)(100 + 50)} \]

\[ 1 = \frac{15K}{11250} \]

\[ K = \frac{750}{11250} \]

\[ K = 750 \quad \text{Ans.} \]
Example 12.68. The open loop transfer function with unity feedback is

\[ G(s) = \frac{K}{s(s+2)(s+20)} \]

Find analytically the value of \( K \) for phase margin = 50\(^\circ\).

**Solution:**

\[ G(s)H(s) = \frac{K}{s(1+0.5s)(1+0.05s)} \]

Put \( s = j\omega \)

\[ G(j\omega)H(j\omega) = \frac{K}{j\omega (1+j0.5\omega)(1+j0.05\omega)} \]

We know that

\[ P.M = 180^\circ + \arg(G(j\omega)H(j\omega)) \]

\[ \frac{G(j\omega)H(j\omega)}{G_H(j\omega)} = -90^\circ - \tan^{-1}0.5\omega - \tan^{-1}0.05\omega \]

\[ 50^\circ = 180^\circ + (-90^\circ - \tan^{-1}0.5\omega - \tan^{-1}0.05\omega) \]

or,

\[ 40^\circ = \tan^{-1}0.5\omega + \tan^{-1}0.05\omega \]

Taking tan on both sides

\[ \tan 40^\circ = \tan (\tan^{-1}0.5\omega + \tan^{-1}0.05\omega) \]

\[ 0.839 = \frac{0.55\omega}{1-0.025\omega^2} \]

or, \[ 0.020975\omega^2 + 0.55\omega - 0.839 = 0 \]

or, \[ \omega = 0.865 \text{ rad/sec.} \]

This is the gain crossover frequency \( \omega_c \)

At gain crossover frequency \( |G(j\omega)H(j\omega)| = 1 \)

\[ \frac{K}{1.445(1+0.7225)(1+j0.07225)} = 1 \]

\[ K = 1.7873 \]

or, \[ K = 71.49 \text{ Ans.} \]

Example 12.69. The open loop transfer function with unity feedback system is

\[ G(s) = \frac{25}{s(s+1)(s+10)} \]

Determine phase margin and gain margin analytically.

**Solution:**

**Step 1:** Calculation of Phase Crossover Frequency

\[ G(j\omega)H(j\omega) = \frac{2.5}{j\omega (1+j\omega)(1+j0.1\omega)} \]

At phase cross-over frequency the angle is -180\(^\circ\).

\[ -180^\circ = -90^\circ - \tan^{-1}0.1\omega - \tan^{-1}\omega \]

or,

\[ 90^\circ = \tan^{-1}0.1\omega + \tan^{-1}\omega \]

Example 12.70. Draw Bode plot of transfer function given in example 12.69 and find P.M. & G.M.

\[ G(j\omega) = \frac{2.5}{j\omega (1+j\omega)(1+j0.1\omega)} \]

**Solution:**

**Step 1:** Draw the magnitude plot

- Plot a line having slope -20 dB/dec. up to 2.5 and mark 1 corner frequency 1 rad/sec. as there is one pole at origin.

**Step 2:** Plot phase angle

- At \( \omega = 1 \) rad/sec, there is a simple pole, so next slope will be -20 \(+ (-20)\) = -40 dB/sec.

- At \( \omega = 10 \) rad/sec. there is a simple pole, so draw a line having slope -20 \(+ (-40)\) = -60 dB/dec. towards \( \infty \).

**Step 3:** Plot phase angle

\[ \phi = -90^\circ - \tan^{-1}0.1\omega - \tan^{-1}\omega \]
Example 12.71. Determine the transfer function of the system using block diagram reduction. The system shown in fig. 12.80.

From Bode Plot
P.M. = 24°
G.M. = 13 db

Fig. 12.79.

Solution: Step 1: Shift the takeoff point after the block $G_d(s)$

Fig. 12.79.

Step 2: Blocks $G_3(s)$ & $G_d(s)$ are in cascade.
Step 3: Solve the inner closed loop

Step 4: Two blocks are in cascade

Step 5: Solve the closed loop

Step 6: Two blocks are in cascade

Step 7: Find \( \frac{C(s)}{R(s)} \)

Example 12.72. Use block reduction technique obtain the transfer function

(i) \( \frac{C(s)}{R(s)} \)

(ii) \( \frac{C(s)}{X(s)} \)

(iii) \( \frac{C(s)}{Y(s)} \)

(iv) Also find total output

Fig. 12.80.

Solution:
(i) When \( R(s) \) acting alone and \( x(s) = 0, Y(s) = 0 \)

(ii) When \( X(s) \) acting alone and \( R(s) = 0, Y(s) = 0 \)
Example 12.73. Using block reduction technique, find the transfer function from each input to the output C. Also total output.

Solution: When $D(s) = 0$

$$C(s) = \frac{G_3(s) G_1(s)}{1 + G_1(s) G_2(s) H_1(s) H_2(s)}$$

(iii) When $Y(s)$ acting alone and $R(s) = 0$, $X(s) = 0$

Solve the closed loop

$$C(s) = \frac{G_1(s) G_2(s) G_3(s) H_1(s) H_2(s)}{1 + G_1(s) G_2(s) G_3(s) H_1(s) H_2(s)}$$

Ans.

(iii) total output: from equations (i), (ii), (iii)

$$C(s) = \frac{G_1(s) G_2(s) G_3(s) R(s) + G_3(s) G_1(s) X(s) - G_3(s) G_1(s) G_2(s) H_1(s) H_2(s) Y(s)}{1 + G_1(s) G_2(s) G_3(s) H_1(s) H_2(s)}$$

Ans.
Example 12.74. Use block diagram reduction methods to obtain the equivalent transfer function from \( R \) to \( C \).

**Solution:** Solve the closed loop of \( G_4(s) \) & \( H_2(s) \)

\[
\frac{C(s)}{R(s)} = \frac{G_4(s)}{1 + G_4(s)H_2(s)} \]

Two blocks are in cascade & solve inner closed loop

\[
\frac{C(s)}{R(s)} = \frac{G_4(s)G_2(s)[G_7(s) + G_3(s)]}{1 + G_4(s)H_2(s)} \]

---

Example 12.75. Simplify the block diagram to obtain transfer function \( \frac{C(s)}{R(s)} \).
Step 6: reduce the inner closed loop

\[ \frac{C(s)}{R(s)} = \frac{G_2 G_3 (G_2 + H_1)}{1 + G_2 H_2 + 2 G_3 H_3 (G_2 + H_1)} \]  Ans.

Example 12.76. Draw the block diagram of a closed loop control system and indicate the following on it:

(i) Plant
(ii) Command input
(iii) Controlled output
(iv) Actuating signal
(v) Feedback element and control element.

Also mention the important features of closed loop control system.

Solution:

\[ \begin{align*}
\text{Command input} & \rightarrow \text{Actuating signal} \\
& \rightarrow \text{Control element} \\
& \rightarrow \text{Plant} \\
& \rightarrow \text{Controlled output} \\
\end{align*} \]

Fig. 12.84.

The important features of closed loop control systems are:

1. Closed loop systems are more reliable
2. They are accurate
3. Optimization is possible
4. Bandwidth of closed loop system is high.
5. A number of variables can be handled simultaneously.
6. Closed loop systems are expensive.
7. Closed loop systems are complicated.

Example 12.77. Write the differential equations and obtain \((f - v)\) force-voltage analogous networks for the system shown below.

\[ \text{Fig. 12.85} \]

The force-voltage analogous circuit is shown in Fig. 12.85(d).
Example 12.78. Derive the transfer function $\frac{E_{in}(s)}{E_{i}(s)}$ of the network shown below.

![Diagram](image)

solution: Refer solved exam. 4.1.3

Example 12.79. Derive the closed loop transfer function $\frac{C(s)}{R(s)} = M(s)$ for the system shown below and find its sensitivity w.r.t $G$ & $H$.

![Diagram](image)

solution: Refer article 1.20 and 1.30.3.

Example 12.80. Define gain margin, phase margin, gain crossover frequency, phase crossover frequency in a polar plot.

solution: Refer article 4.6

Example 12.81. State the rules for construction of root loci of $G(s)H(s)$. Find the breakaway points of $G(s)H(s) = \frac{k}{s(s+4)(s^2+4s+20)}$.

solution: For construction of root loci refer article 5.8 and for breakaway point refer example 5.33.

Example 12.82. A unity feedback system has open loop transfer function $G(s) = \frac{k}{s(0.1s+1)(0.2s+1)}$. The system is required to satisfy the following performance specifications $K_p = 30$, $\omega_n \geq 40$ bandwidth $\omega_b = 5$ rad/sec. Design a lag compensator.

solution: Refer example 6.4

Example 12.83. Obtain the state model of the system whose transfer function is given by $T(s) = \frac{s^3 + 3s + 3}{s^3 + 2s^2 + 3s + 1}$ and hence, find its state transition matrix.

![Diagram](image)

solution: Let $T(s) = \frac{C(s)}{R(s)} = \frac{s^3 + 3s + 3}{s^3 + 2s^2 + 3s + 1}$.

Divide numerator & denominator by $s^3$.

$C(s) = \frac{s^3 + 3s^2 + 3s + 3}{s^3 + 2s^2 + 3s + 1}$.

The above equations are controllable canonical form.

Example 12.84. For the state matrix $A = \begin{bmatrix} 0 & 1 \\ -8 & -6 \end{bmatrix}$, find $e^{At}$ by any two methods.
Solution: First method: Use of diagonal

\[
A = \begin{bmatrix} 0 & 1 \\ -8 & -6 \end{bmatrix}
\]

\[
\lambda I - A = \begin{bmatrix} 0 & 1 \\ -8 & -6 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} = \begin{bmatrix} \lambda & 1 \\ -8 & -6 - \lambda \end{bmatrix}
\]

\[
|\lambda I - A| = (\lambda^2 + 6\lambda + 8) = (\lambda + 2)(\lambda + 4) = 0
\]

\[
\lambda_1 = -2, \; \lambda_2 = -4
\]

Let the eigen vector \( p_1 = \begin{bmatrix} p_{11} \\ p_{21} \end{bmatrix} \) & \( p_2 = \begin{bmatrix} p_{12} \\ p_{22} \end{bmatrix} \)

Put \( \lambda = -2 \) in equation (i)

\[
\begin{bmatrix} 2 & 1 \\ 8 & 4 \end{bmatrix} \begin{bmatrix} p_{11} \\ p_{21} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}
\]

\[-2p_{11} - p_{21} = 0
\]

Let \( p_{11} = 1 \) then \( p_{21} = -2
\]

\[
p_1 = \begin{bmatrix} 1 \\ -2 \end{bmatrix}
\]

Put \( \lambda = -4
\]

\[
\begin{bmatrix} -4 & 1 \\ 8 & 2 \end{bmatrix} \begin{bmatrix} p_{12} \\ p_{22} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}
\]

\[-4p_{12} - p_{22} = 0
\]

Let \( p_{12} = 1 \) then \( p_{22} = -4
\]

\[
p_2 = \begin{bmatrix} 1 \\ -4 \end{bmatrix}
\]

\[
M = [p_1 \; p_2] = \begin{bmatrix} 1 & 1 \\ -2 & -4 \end{bmatrix}
\]

\[
M^{-1} = \begin{bmatrix} 2 & 0.5 \\ -1 & -0.5 \end{bmatrix} \quad \text{&} \quad \lambda = \begin{bmatrix} -2 & 0 \\ 0 & -4 \end{bmatrix}
\]

\[
e^{\lambda t} = Me^{\lambda t} M^{-1} = \begin{bmatrix} 1 & 1 \\ -2 & -4 \end{bmatrix} \begin{bmatrix} e^{-2t} & 0 \\ 0 & e^{-4t} \end{bmatrix} \begin{bmatrix} 2 & 0.5 \\ -1 & -0.5 \end{bmatrix}
\]

\[
e^{\lambda t} = \begin{bmatrix} 2e^{-2t} - e^{-4t} & 0.5e^{-2t} - 0.5e^{-4t} \\ -4e^{-2t} + 4e^{-4t} & -e^{-2t} + 2e^{-4t} \end{bmatrix}
\]

Example 12.85: Derive the state variable model for the system shown below

\[\text{Frictionless wheel}
\]

\[\text{Fig. 12.87.}
\]

\[\text{Schematic: Mechanical equivalent diagram}
\]

\[\text{at node (i)}
\]

\[M_1 \ddot{x}_1 + D \dot{x}_1 + k(x_1 - x_2) = 0
\]

\[\text{at node (ii)}
\]

\[f(t) = M_2 \ddot{x}_2 + k(x_2 - x_1)
\]
or, \( M_1 \ddot{x}_1 + D \dot{x}_1 + k x_1 - k x_2 = 0 \)

Select the state variable as

\[
\begin{align*}
\dot{x}_1 &= z_1 \\
\dot{x}_2 &= z_2 \\
\dot{x}_3 &= z_3 \\
\dot{x}_4 &= z_4
\end{align*}
\]

\[
\begin{align*}
\dot{z}_1 &= 0 \\
\dot{z}_2 &= k/M_1 z_1 - D/M_1 z_2 + k/M_2 z_3 \\
\dot{z}_3 &= 0 \\
\dot{z}_4 &= k/M_2 z_1 - k/M_2 z_4
\end{align*}
\]

State equations are

\[
\begin{align*}
\dot{z}_2 &= \frac{k}{M_1} z_1 - \frac{D}{M_1} z_2 + \frac{k}{M_2} z_3 \\
\dot{z}_3 &= 0 \\
\dot{z}_4 &= \frac{k}{M_2} z_1 - \frac{k}{M_2} z_4
\end{align*}
\]

Select the displacement of mass \( M_1 \) & \( M_2 \) as output variables.

\[
\begin{align*}
y_1 &= x_1 \\
y_2 &= x_2 \\
y_3 &= x_3 \\
y_4 &= x_4
\end{align*}
\]

\[\text{Ans.}\]

Example 12.86. Define natural frequency of oscillation \( \omega_n \) and damping ratio of second order system. Find the nature of damping for the following systems.

\[
\begin{align*}
\text{(i)} & \quad R(s) \xrightarrow{\frac{12}{s^2 + 8s + 12}} C(s) \\
\text{(ii)} & \quad R(s) \xrightarrow{\frac{16}{s^2 + 8s + 16}} C(s) \\
\text{(iii)} & \quad R(s) \xrightarrow{\frac{20}{s^2 + 8s + 20}} C(s)
\end{align*}
\]

\[\text{(i)} \quad \frac{C(s)}{R(s)} = \frac{12}{s^2 + 8s + 12} \]

\[\text{(ii)} \quad \frac{C(s)}{R(s)} = \frac{16}{s^2 + 8s + 16} \]

\[\text{(iii)} \quad \frac{C(s)}{R(s)} = \frac{20}{s^2 + 8s + 20} \]

\[\omega_n^2 = 16 \quad \omega_n = 4 \text{ rad/sec.} \]

\[2\zeta \omega_n = 8 \quad \zeta = 1 \]

Since, \( \zeta = 1 \), the system is critically damped.

\[\omega_n^2 = 20 \quad \omega_n = 4.472 \text{ rad/sec.} \]

\[2\zeta \omega_n = 8 \quad \zeta = 0.894 \]

Since, \( \zeta < 1 \), the system is underdamped.

Example 12.87. Find the damping factor \( \zeta \), natural frequency \( \omega_n \), peak time \( T_p \) and percentage overshoot for the system with

\[G(s) = \frac{1}{9s^2 + 2s + 58} \]
Example 12.88. A closed loop servo system is represented by the differential equation
\[ \frac{d^2 C}{dt^2} + 8 \frac{dC}{dt} + 64 C = 64e \]
where \( C \) is the displacement of the output shaft, \( r \) is the displacement of the input shaft and \( e = r - c \). Determine: (i) damping ratio (ii) damped natural frequency (iii) % M_p

Solution:
\[ \frac{d^2 C}{dt^2} + 8 \frac{dC}{dt} + 64 C = 64 (r - C) \]
\[ s^2 C(s) + 8sC(s) = 64 [R(s) - C(s)] \]
\[ C(s) \frac{s^2 + 8s + 64}{s^2 + 8s + 64} = 64 R(s) \]
\[ \frac{C(s)}{R(s)} = \frac{64}{s^2 + 8s + 64} \]
\[ \omega_n^2 = \frac{64}{s^2 + 8s + 64} \]
\[ \omega_n = 8 \text{ rad/sec.} \]
\[ 2\zeta \omega_n = 8 \]
\[ \zeta = 0.5 \]
\[ \% M_p = 100 \frac{\zeta}{\sqrt{1 - \zeta^2}} = 16.3 \% \text{ Ans.} \]

Example 12.89. Sketch the root locus for the system having
\[ G(s) = \frac{k}{(s^2 + 2s + 2)(s^2 + 2s + 5)} \]

Solution:
Step 1: Root locus is symmetrical about real axis.
Step 2: No. of poles = 4 at \( s = -1 \pm j \) and \( s = -1 \pm 2 \) No. of zeros = 0
Step 3: No. of root loci = No. of poles (\( \because p > z \)) = 4
Step 4: Centroid of asymptotes
\[ \sigma_A = - \frac{1}{2} \left( -1 - 1 - 1 + 2 \right) = 0 \]
\[ \sigma_A = -1 \]
Step 5: Angle of asymptotes
\[ \phi = \frac{2k + 1}{p - z} \times 180^\circ \]
For \( k = 0, 1, 2, 3 \)
\[ \phi_1 = 45^\circ, \phi_2 = 135^\circ, \phi_3 = 225^\circ, \phi_4 = 315^\circ \]
Step 6: Calculation of Breakaway point

The characteristic equation: \[ 1 + G(s) H(s) = 0 \]
\[ 1 + \frac{k}{(s^2 + 2s + 2)(s^2 + 2s + 5)} = 0 \]
\[ k = (s^4 + 4s^3 + 11s^2 + 14s + 10) \]
\[ \frac{dk}{ds} = (4s^3 + 12s^2 + 22s + 14) \]
\[ \frac{dk}{ds} = (s + 1) (2s^2 + 4s + 7) \]
\[ \frac{dk}{ds} = (s + 1) (2s^2 + 4s + 7) \]
\[ \text{Roots of} \frac{dk}{ds} \text{ are} \]
\[ s = -1, -1 \pm 1.58 \]
\[ s = -1 \text{ lies on real axis, this does not belong to root locus so this is not the breakaway point.} \]
\[ \text{At} s = -1 \pm 1.58 \text{ angle criterion is satisfied hence it is the breakaway point.} \]
Step 7: Real axis is not the part of root locus.
Step 8: Point of intersection with imaginary axis.
The characteristic equation is
\[ s^4 + 4s^3 + 11s^2 + 14s + (10 + k) = 0 \]
\[ s^4 \]
\[ s^3 \]
\[ s^2 \]
\[ s^1 \]
\[ s^0 \]
\[ 1 \]
\[ 4 \]
\[ 10 + k \]
\[ 65 - 4k \]
\[ 7.5 \]
\[ 0 \]
\[ 10 + k \]
For roots on imaginary axis
\[ 65 - 4k = 0 \]
or
\[ k = 16.25 \]
Auxiliary equation
\[ 7.5s^2 + (10 + k) = 0 \]
\[ 7.5s^2 + (10 + 16.25) = 0 \]
\[ s = \pm 1.87 \]
Step 9: Calculation of angle of departure
\[ \Phi_{\theta} \bigg|_{s = -1 \pm 1.87} = 90^\circ \]
\[ \phi = 180 - (90^\circ + 90^\circ) = -90^\circ \]
\[ \Phi_{\theta} \bigg|_{s = -1 \pm 1.87} = 180 - (90^\circ + 90^\circ + 90^\circ) = +90^\circ \]
\[ \text{angle of departure} \bigg|_{s = -1 \pm 1.87} = \pm 90^\circ \]
The complete root locus is shown in fig. 12.88.
Example 12.90. Obtain the closed loop transfer function (in Example 12.89) such that the system exhibits oscillatory response.

Solution: Closed loop transfer for any gain $k$ is

$$C(s) = \frac{k \text{closed loop zero factors}}{R(s) \text{closed loop pole factors}}$$

The number of closed loop pole factors is same as the no. of root locus branches i.e. each closed loop pole factor corresponds to each root locus branch.

Closed loop zeros are same as the open loop zeros. There are four root locus branches where $k$ is varying from zero to infinity. So, there must be four closed loop poles. Here we have to determine the closed loop transfer function for oscillatory response i.e. for that value of $k$ for which poles lie on imaginary axis i.e. $k = 16.25$. Now identify four points on root locus branches above $k = 16.25$ (either by trial & error or logically). These will be the four closed loop poles.

From the root locus shown in fig. 12.88 it is clear that the two closed loop poles on the two root locus branches for $k = 16.25$ are $s = ±j1.87$ & $s = -j1.87$. Since root locus is symmetrical w.r.t. vertical line $s = 1$, the remaining two closed loop poles on other two root locus branches can be obtained by projecting points $s = ±j1.87$ on the other two root locus branches as shown at point $1 & 2$. Also by magnitude criterion for $k = 16.25$ the corresponding closed loop poles are $s = -2 ±j1.87$.

Since there are no open loop zeros, there will be no closed loop zeros. Hence required closed loop transfer function for $k = 16.25$ is

$$C(s) = \frac{16.25}{(s-j1.87)(s+j1.87)(s+2-j1.87)(s+2+j1.87)}$$

Example 12.91. For a rotational system shown in fig. 12.89 draw electrical analogous circuit based on $T$-i analogy.

Solution: Draw the mechanical equivalent circuit.
Now electrical analogous circuit can be drawn as

![Electrical Circuit Diagram](image)

**Example 12.92.** For the system shown in fig 12.90, find $s^T_C$.

![System Diagram](image)

**Solution:** Overall transfer function $T(s)$ will be

$$T(s) = \frac{G_1 G_2}{1 + G_2}$$

$$s^T_C = \frac{G_1}{1 + G_2} \cdot \frac{\partial T(s)}{\partial G_1}$$

$$\frac{\partial T(s)}{\partial G_1} = \frac{(1 + G_2)C_2 - G_1 C_0}{(1 + G_2)^2} = \frac{G_1}{1 + G_2}$$

$$s^T_C = \frac{G_1 G_2}{1 + G_2} \cdot \frac{C_2}{1 + G_2}$$

$$\boxed{s^T_C = 1}$$

**Ans.**

**Example 12.93.** If bandwidth increases, then what will be the effect on damping ratio and rise time.

**Solution:** If bandwidth increases, the rise time will be small i.e. the system response will be fast. So, if bandwidth increases then rise time & damping ratio will decrease.

**Example 12.94.** Draw the polar plot for a system given by

$$G(s)H(s) = \frac{100}{s(s+2)(s+4)(s+8)}$$

Find whether the system is stable or not. Also find out P.M, G.M, phase cross-over frequency and gain crossover frequency.

**Solution:** Put $s = j\omega$

$$G(j\omega)H(j\omega) = \frac{100}{j\omega (2 + j\omega)(4 + j\omega)(8 + j\omega)}$$

$$|G(j\omega)H(j\omega)| = \frac{100}{\omega^2 + \omega^4 + \omega^2 + \omega^2 + \omega^2 + \omega^2 + \omega^2 + \omega^2 + \omega^2 + \omega^2}$$

$$\angle G(j\omega)H(j\omega) = -90^\circ - \tan^{-1}\left(\frac{1}{2}\right) - \tan^{-1}\left(\frac{1}{4}\right) - \tan^{-1}\left(\frac{1}{8}\right)$$

A rough sketch of the polar plot is shown in fig 12.91.

For stability determine the value of $\omega$ i.e. distance from origin to 'a' the point 'a' is called phase cross-over point and frequency at this point is called phase cross-over frequency.

At point 'a' $\phi = -180^\circ$

$$\omega = 2.2 \text{ rad/sec.}$$

Hence $\omega = 2.2$ rad/sec. **Ans.**

This is the phase cross-over frequency. Now find magnitude at this frequency.

$$|G(j\omega)H(j\omega)|_{a=2.2} = \frac{100}{\sqrt{(2.2)^2 + (2.2)^2 + (2.2)^2 + (2.2)^2 + (2.2)^2 + (2.2)^2 + (2.2)^2 + (2.2)^2 + (2.2)^2 + (2.2)^2}} = 0.403$$

Hence, $x < 1$, the system is stable **Ans.**

G.M. = $\frac{1}{x} = \frac{1}{0.403} = 2.47$ **Ans.**

Now calculate gain cross-over frequency. At this frequency $|G\omega| = 1$

$$\omega = 1.2 \text{ rad/sec.}$$

This is the gain cross-over frequency = 1.2 rad/sec. **Ans.**

$$\phi = -90^\circ - \tan^{-1}\left(\frac{1.2}{2}\right) - \tan^{-1}\left(\frac{1.2}{4}\right) - \tan^{-1}\left(\frac{1.2}{8}\right)$$

$$= -146.18^\circ$$

P.M = $180^\circ + (-146.18^\circ) = 33.82^\circ$ **Ans.**
Example 12.95. Obtain the transfer function of the given state equation

\[
\begin{bmatrix}
    x_1 \\
    x_2 \\
    x_3 
\end{bmatrix} =
\begin{bmatrix}
    0 & 1 & 0 \\
    -1 & -1 & 0 \\
    1 & 0 & 0 
\end{bmatrix}
\begin{bmatrix}
    x_1 \\
    x_2 \\
    x_3 
\end{bmatrix} + \begin{bmatrix}
    1 
\end{bmatrix} u
\]

\[y = \begin{bmatrix}
    0 & 0 & 1 
\end{bmatrix}
\begin{bmatrix}
    x_1 \\
    x_2 \\
    x_3 
\end{bmatrix}
\]

Solution: Here,

\[A = \begin{bmatrix}
    0 & 1 & 0 \\
    -1 & -1 & 0 \\
    1 & 0 & 0 
\end{bmatrix}, \quad B = \begin{bmatrix}
    0 
\end{bmatrix}, \quad C = \begin{bmatrix}
    0 & 0 & 1 
\end{bmatrix}
\]

\[SL - A = \begin{bmatrix}
    s & -1 & 0 \\
    1 & s + 1 & 0 \\
    -1 & 0 & s 
\end{bmatrix}
\]

\[|SL - A| = s^3 + s^2 + s
\]

\[\text{Adj} (SL - A) = \begin{bmatrix}
    s^2 + s & s & 0 \\
    -s & s^2 + s + 1 & 0 \\
    s + 1 & 1 & s^2 + s + 1 
\end{bmatrix}
\]

\[\left(\frac{\text{Adj} (sI - A)}{sI - A}\right) = \begin{bmatrix}
    s^2 + s + 1 & s^2 + s + 1 & 0 \\
    -s & s^2 + s + 1 & s^2 + s + 1 \\
    s + 1 & 1 & s^2 + s + 1 
\end{bmatrix}
\]

\[C^T[A]^{-1}B = \begin{bmatrix}
    0 & 0 
\end{bmatrix}
\begin{bmatrix}
    x_1 \\
    x_2 \\
    x_3 
\end{bmatrix}
\]

This is the required transfer function.

\[C(s) = Q(s) [10.4s^{-1} + 47s^2 + 160s^{-3}]
\]

Example 12.97. Obtain the time response of the following system

\[\dot{x} = \begin{bmatrix}
    0 & 1 \\
    -6 & -5 
\end{bmatrix} x + \begin{bmatrix}
    0 
\end{bmatrix} u
\]

\[y = \begin{bmatrix}
    1 & 0 
\end{bmatrix} x
\]

Where \(u(t)\) is a unit step input and initial conditions \(x_1(0) = 0\) and \(x_2(0) = 0\)

Solution: Here,

\[A = \begin{bmatrix}
    0 & 1 \\
    -6 & -5 
\end{bmatrix}, \quad B = \begin{bmatrix}
    0 
\end{bmatrix}
\]

\[SL - A = \begin{bmatrix}
    s & 0 \\
    0 & s 
\end{bmatrix} = \begin{bmatrix}
    s & -1 \\
    6 & s + 5 
\end{bmatrix}
\]

\[sI - A = s^2 + 6s + 6 = (s + 2)(s + 3)
\]

cofactors of \((sI - A)\) = \[\begin{bmatrix}
    s + 5 & -6 \\
    1 & 5 
\end{bmatrix}
\]
\(\text{Adj of } (sl-A) = \begin{bmatrix} s+5 & 1 \\ 6 & 5 \end{bmatrix}\)

\([sl-A]^{-1} = \begin{bmatrix} \frac{s+5}{6} & \frac{1}{s+3} \\ -\frac{1}{s+2} & \frac{s}{s+3} \end{bmatrix}\)

or,

\([sl-A]^{-1} = \begin{bmatrix} \frac{3}{s+2} & \frac{2}{s+2} & \frac{1}{s+3} \\ \frac{s+2}{s+2} & \frac{-2}{s+2} & \frac{-2}{s+3} \end{bmatrix}\)

\(e^{-t} [sl-A]^{-1} = \begin{bmatrix} 3e^{-2t} - 2e^{-3t} & e^{-2t} - e^{-3t} \\ -6e^{-2t} + 6e^{-3t} & -2e^{-2t} + 3e^{-3t} \end{bmatrix}\)

\[x(t) = e^{sl}x(0) + \int_{0}^{t} e^{s(t-\tau)} Bu(\tau) d\tau\]

\[x(t) = \begin{bmatrix} 3e^{-2t} - 2e^{-3t} & e^{-2t} - e^{-3t} \\ -6e^{-2t} + 6e^{-3t} & -2e^{-2t} + 3e^{-3t} \end{bmatrix} x(0) + \int_{0}^{t} \begin{bmatrix} 3e^{-2(t-\tau)} - 2e^{-3(t-\tau)} \\ -6e^{-2(t-\tau)} + 6e^{-3(t-\tau)} \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} d\tau\]

\[x(t) = \int_{0}^{t} e^{-2(t-\tau)} - e^{-3(t-\tau)} d\tau\]

\[x_{1}(t) = \int_{0}^{t} e^{-2(t-\tau)} - e^{-3(t-\tau)} d\tau\]

\[x_{2}(t) = \int_{0}^{t} e^{-2(t-\tau)} - e^{-3(t-\tau)} d\tau\]

\[x_{1}(t) = \frac{1}{6} - \frac{1}{2} e^{-2t} - \frac{1}{3} e^{-3t}\] Ans.

\[x_{2}(t) = e^{3t} - e^{3t}\] Ans.
or,
\[
\frac{Y(s)}{u(s)} = C(sI - A)^{-1} B + D = C \frac{\text{Adj}(sI - A)}{sI - A} B + D
\]

The denominator \( sI - A \) is the characteristic equation. Hence we can calculate the poles
\[
sI - A = \begin{bmatrix} s & 0 \\ 0 & s \end{bmatrix} - \begin{bmatrix} 0 & 1 \\ -20 & -9 \end{bmatrix} = \begin{bmatrix} s - 1 \\ 20 & s + 9 \end{bmatrix}
\]
\[
|sI - A| = s(s + 9) + 20
\]
\[
= s^2 + 9s + 20 = 0
\]
or
\[
(s + 5)(s + 4) = 0
\]
\[
s = -5, -4
\]

Hence, the poles are \(-5, -4\). The cofactors of \( sI - A \) can be calculated as
\[
(-1)^1 \cdot 1 (s + 9) = s + 9
\]
\[
(-1)^1 \cdot 2 (20) = -20
\]
\[
(-1)^2 \cdot 1 (-1) = 1
\]
\[
(-1)^2 \cdot 2 (s) = s
\]
\[
\therefore \text{cofactor of } (sI - A) = \begin{bmatrix} s + 9 & -20 \\ 1 & s \end{bmatrix}
\]

Adj of
\[
\text{Adj}(sI - A) = \begin{bmatrix} s + 9 & 1 \\ -20 & s \end{bmatrix}
\]

Now equate the equation (vi) to zero we get
\[
C \text{Adj}(sI - A)B + |sI - A| D = 0
\]
\[
\begin{bmatrix} -17 \\ -5 \end{bmatrix} \begin{bmatrix} s + 9 & 1 \\ -20 & s \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} + (s^2 + 9s + 20) 1 = 0
\]
\[
-17 - 5s + s^2 + 9s + 20 = 0
\]
\[
s^2 + 4s + 3 = 0
\]
\[
(s + 3)(s + 1) = 0
\]
\[
\therefore \text{zeros are at } s = -1, -3
\]

Note: In this Example value of \( D \) is given.

Example 12.100. A close loop transfer function of a unity feedback control system is
\[
\frac{C(s)}{R(s)} = \frac{20s^2}{(s + 1)(s + 3)(s + 5)}
\]

Example 12.102. Draw the root locus diagram for the following control system and calculate the break in and breakaway points.

Example 12.101. A unity feedback control system has a forward transfer function \( \frac{25}{s(s + 6)} \). Find the resonance peak and the corresponding frequency for the closed loop frequency response. Derive the formula used.

Solution:
\[
\frac{C(s)}{R(s)} = \frac{25}{s(s + 6)}
\]

\[
\omega_n^2 = 25 \therefore \omega_n = 5 \text{ rad/sec.}
\]
\[
2\zeta\omega_n = 6 \therefore \zeta = 0.6
\]
\[
\omega_r = \omega_n \sqrt{1 - 2\zeta^2} = 5 \sqrt{1 - 2(0.6)^2} = 2.65 \text{ rad/sec. Ans.}
\]
\[
M = \frac{1}{2\zeta \sqrt{1 - \zeta^2}} = \frac{1}{2 \times 0.6 \sqrt{1 - (0.6)^2}} = 1.042 \text{ Ans.}
\]

For derivation refer article no. 4.15.
Solution: From the Fig. 12.93
\[ G(s) H(s) = \frac{K(s+2)(s+3)}{s(s+1)} \]

Step 1: Plot the Poles and Zeros
Poles: \( s_1 = 0, s_2 = -1 \)
Zeros: \( s_3 = -2, s_4 = -3 \)

Step 2:
- No. of poles = 2
- No. of zeros = 2

Since, no. of poles = no. of zeros, therefore, there will be no asymptotes.

Step 3: Characteristic equation
\[ 1 + \frac{k(s+2)(s+3)}{s(s+1)} = 0 \]
\[ k = \frac{s^2 + s}{s^2 + 5s + 6} \]

For break points
\[ \frac{dk}{ds} = \frac{d}{ds} \left[ \frac{s^2 + s}{s^2 + 5s + 6} \right] = 0 \]
\[ 2s + 1 = 2s + 5 = 0 \]
\[ s = -0.634, -2.366 \]

As, \( s = -0.634 \) lies between two poles \( s = 0 \) and \( s = -1 \), therefore it is a breakaway point.
\( s = -2.366 \) lies between two zeros at \( s = -2 \) and \( s = -3 \), therefore it is a break-in point.

Hence, root locus is a circle shown in Fig. 12.94 with centre at \((-1.5, 0)\) and passing through breakaway points.

Example 12.103. The system shown in Fig. 12.95 consists of a unity feedback loop containing a simple rate feed-back loop.
(i) Without any rate feedback \((b = 0)\), determine the damping factor, natural frequency, overshoot of the system to a unit step input and steady state error resulting from a unit ramp input.

(ii) Determine the rate feedback constant which will increase the equivalent damping factor of the system to 0.8. Determine the overshoot of the system in this case to a unit step input and steady state error resulting from a unit ramp input.

Solution:

When \( b = 0 \)
\[ C(s) = \frac{16}{s(s+4)} \]
\[ R(s) = 1 + \frac{16}{s(s+4)} = \frac{s^2 + 4s + 16}{s^2 + 4s + 16} \]

Compare with
\[ C(s) = \frac{w_n^2}{s^2 + 2zw_n + w_n^2} \]
\[ w_n^2 = 16 \quad \therefore \quad w_n = 4 \text{ rad/sec.} \]
\[ 2zw_n = 4 \quad \therefore \quad z = 0.5 \]

Resonant frequency
\[ \omega_R = \frac{\omega_n}{\sqrt{1 - z^2}} \]
\[ \omega_R = 4 \sqrt{1 - 0.5^2} = 2.83 \text{ rad/sec.} \quad \text{Ans.} \]
\[ M_p = \frac{0.8}{\sqrt{1 - z^2}} \]
\[ M_p = 100 \quad \therefore \quad M_p = 100 = 16.32\% \quad \text{Ans.} \]

Steady state error
\[ e_{ss} = \lim_{s \to 0} \frac{R(s)}{1 + G(s)H(s)} \]
\[ H(s) = 1 \]
\[ \frac{1}{s} \]
\[ e_{ss} = \lim_{s \to 0} \frac{R(s)}{1 + G(s)H(s)} = \lim_{s \to 0} \frac{1/s^2}{1 + \frac{16}{s(s+4)}} = \frac{1}{4} \]

\[ e_{ss} = 0.25 \text{ Ans.} \]

(ii) Consider feedback.
\[ C(s) = \frac{16}{s^2 + (4 + 16\omega_n)s + 16} \]

Compare with
\[ C(s) = \frac{w_n^2}{s^2 + 2zw_n + w_n^2} \]
Example 12.104. Find the transfer function and draw the block diagram of the circuit shown in Fig. 12.96.
Solution: Draw the transform network.

**KCL at node (1)**
\[
\frac{V_1(s)}{V(s)} - \frac{V(s) - V_2(s)}{1} = \frac{V(s)}{s}
\]

or
\[
V(s) = \frac{V_2(s)}{s} + \frac{s + 1}{s}
\]

**KCL at node (2)**
\[
\frac{V(s) - V_2(s)}{1} = \frac{V_2(s)}{s}
\]

Equate the equations (1) and (2)
\[
\frac{V(s)}{s} = \frac{s + 1}{s} + \frac{s^2}{s^2 + 3s + 1}
\]

For block diagram
\[
\frac{V_1(s) - V(s)}{1} = \frac{V(s) - V_2(s)}{1}
\]
\[
V_2(s) = s I_2(s)
\]

Example 12.105. Find the number of roots with positive real part, negative real part and zero real part of the following equations.

(i) \(s^2 + 1 = 0\)

(ii) \(s^3 + 3s^2 + 5s^2 + 9s^2 + 17s^4 + 33s^3 + 31s^2 + 27s + 18 = 0\)

Solution (i):

\[
s^2 + 1 = 0
\]

\[
s = \cos \frac{\pi}{2} + j \sin \frac{\pi}{2}
\]

\[
-1 = \cos \frac{\pi}{2} + j \sin \frac{\pi}{2}
\]

For \(n = 0, s_1 = \cos 30^\circ + j \sin 30^\circ = 0.866 + j0.5\)

\(n = 1, s_2 = \cos 90^\circ + j \sin 90^\circ = 0 + j1\)

\(n = 2, s_3 = \cos 150^\circ + j \sin 150^\circ = -0.866 + j0.5\)

\(n = 3, s_4 = \cos 210^\circ + j \sin 210^\circ = -0.866 - j0.5\)

\(n = 4, s_5 = \cos 270^\circ + j \sin 270^\circ = 0 - j1\)

\(n = 5, s_6 = \cos 330^\circ + j \sin 330^\circ = 0.866 - j0.5\)

No. of roots with positive real part = 2

No. of roots with negative real parts = 2

No. of roots with zero real parts = 2
(b) \( s^5 + 3s^4 + 5s^3 + 9s^2 + 17s + 33s + 31s^2 + 27s + 18 = 0 \)

\[
\begin{array}{cccccc}
    & s^5 & 1 & 5 & 31 & 18 \\
    s^4 & 3 & 9 & 33 & 27 & \\
    s^3 & 2 & 6 & 22 & \\
    s^2 & 0 & 0 & \\
\end{array}
\]

Since one row is zero, therefore auxiliary equation will be

\[
A(s) = 2s^6 + 6s^4 + 22s^2 + 18
\]

or

\[
A(s) = s^6 + 3s^4 + 11s^2 + 9
\]

The auxiliary equation is also a factor of original characteristic equation, therefore characteristic equation can also be written as

\[
(s^2 + 3s + 2)(s^4 + 3s^2 + 11s^2 + 9) = 0
\]

Consider \( s^2 + 3s + 2 = 0 \)

the roots are \( s_1 = -1 + j0, s_2 = -2 \)

Now consider \( s^4 + 3s^2 + 11s^2 + 9 = 0 \)

put \( s^2 = x \)

\[
\begin{align*}
    x^3 + 3s^2 + 11x + 9 &= 0 \\
    (x + 1)(x^2 + x + 9) &= 0 \\
    x + 1 &= 0
\end{align*}
\]

or \( s^2 + 1 = 0 \)

\[ s_3 = \pm j1 \]

\[ s_4 = -j1 \]

or \( s^4 + 2x^2 + 9 = 0 \)

or \( (s^2 + 2s + 3)(s^2 - 2s + 3) = 0 \)

\( s^2 + 2s + 3 = 0 \), the roots will be

\[ s_5 = -1 + j\sqrt{2} \]

\[ s_6 = -1 - j\sqrt{2} \]

\( s^2 - 2s + 3 = 0 \), the roots will be

\[ s_7 = 1 + j\sqrt{2} \]

\[ s_8 = 1 - j\sqrt{2} \]

:. Required roots are

\[ s_1 = -1 + j0 \]

\[ s_2 = -2 + j0 \]

\[ s_3 = 0 + j1 \]

\[ s_4 = 0 - j1 \]

\[ s_5 = -1 + j\sqrt{2} \]

\[ s_6 = -1 - j\sqrt{2} \]

\[ s_7 = 1 + j\sqrt{2} \]

\[ s_8 = 1 - j\sqrt{2} \]

\[ s_9 = 0 \]

No. of roots with positive real part = 2

No. of roots with negative real parts = 4

No. of roots with zero real parts = 2.

Example 12.106. For the given network, the output is \( V_1 \) and input is \( V_i \)

(a) Write an equation for \( V_2 \) in terms of \( V_1, R_1 \) and \( R_2 \) which yields an open loop system.

(b) Write an equation for \( V_2 \) which yields a closed loop system.

\[
\begin{align*}
V_2 &= iR_2 \\
V_2 &= V_1 - \frac{R_2}{R_1 + R_2}
\end{align*}
\]

Equation (3) is the required equation. Equation (3) can also be obtained directly by apply voltage divider rule.

\[
\begin{align*}
V_2 &= \frac{V_1 - V_2}{R_1} \\
V_2 &= \frac{R_2}{R_1 + R_2}
\end{align*}
\]

Equation (5) is the required equation. Ans.

Example 12.107. Simplify the signal flow graph.

Solution: (a)

\[
\begin{align*}
X_3 &= X_1 + X_2 \\
X_1 &= X_2 + X_3
\end{align*}
\]

We have \( X_2 = BX_1 \) and \( X_1 = X_2A \)

\[ X_2 = BAX_2 \] or \( X_1 = ABX_1 \)

Yields Fig. (a) and (b).

(b)

\[
\begin{align*}
X_2 &= X_1A + BX_2 \\
\text{or} \hspace{1cm} X_2 &= \frac{AX_1}{1 - B}
\end{align*}
\]

We have

\[
\begin{align*}
X_2 &= BX_1 \text{ and } X_1 = X_2A \\
X_2 &= BAX_2 \text{ or } X_1 = ABX_1
\end{align*}
\]

Yields Fig. (a) and (b).

\[
\begin{align*}
X_2 &= X_1A + BX_2 \\
\text{or} \hspace{1cm} X_2 &= \frac{AX_1}{1 - B}
\end{align*}
\]
Example 12.108. How many roots of the following polynomial are in the RHS plane and on jω-axis

\[ P(s) = s^5 + 2s^4 + 2s^3 + 4s^2 + s + 2 \]

Solution:

\[ s^5 \quad 1 \quad 2 \quad 1 \]
\[ s^4 \quad 2 \quad 4 \quad 2 \]
\[ s^3 \quad 8 \quad 8 \]
\[ s^2 \quad 2 \quad 2 \]
\[ s \quad 0 \]
\[ 1 \]

Third row having all elements zero.

Auxiliary equation

\[ A(s) = 2s^4 + 4s^2 + 2 \]

\[ \frac{dA(s)}{ds} = 8s^3 + 8s \]

\[ s^5 \quad 1 \quad 2 \quad 1 \]
\[ s^4 \quad 2 \quad 4 \quad 2 \]
\[ s^3 \quad 8 \quad 8 \]
\[ s^2 \quad 2 \quad 2 \]
\[ s \quad 0 \]
\[ 1 \]

Again (V) row having all zero elements

\[ A(s) = 2s^2 + 2 \]

\[ \frac{dA(s)}{ds} = 4s \]

Since no sign change in first column therefore

(a) no roots lies on right half of s-plane

(b) since auxiliary equation contains roots on jω-axis therefore from equation (1) 4 roots lies on jω-axis

(c) Now divide the given polynomial by equation (1)
Modal matrix:
\[
\begin{bmatrix}
1 & 1 & 1 \\
-2 & -5 & -6 \\
4 & 25 & 36
\end{bmatrix}
\]

(ii) From the given circuit:
For mesh (1):
\[
L_1 \frac{dl_1}{dt} + R_l l_1 + e_r - V_c = 0
\]
\[
\frac{dl_1}{dt} = -\frac{R_l}{L_1} l_1 - \frac{e_r}{L_1} + \frac{1}{L_1} V_c
\]
For mesh (2):
\[
\frac{dl_2}{dt} = -\frac{R_s}{L_1} l_2 + \frac{1}{L_2} V_c
\]
KCL at node (2):
\[
i_1 + i_2 + C \frac{dV_c}{dt} = 0
\]
\[
\frac{dV_c}{dt} = \frac{-1}{C} l_1 - \frac{1}{C} l_2
\]
\[
\begin{bmatrix}
i_1 \\
i_2 \\
V_c
\end{bmatrix} = \begin{bmatrix}
\frac{-R_s}{L_1} & 0 & \frac{1}{L_1} \\
0 & \frac{-R_s}{L_2} & \frac{1}{L_2} \\
\frac{-1}{C} & \frac{-1}{C} & 0
\end{bmatrix} \begin{bmatrix}
i_1 \\
i_2 \\
V_c
\end{bmatrix} + \begin{bmatrix}
e_r
\end{bmatrix}
\]
Answ. 

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